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ON THE ITERATED SUSPENSION

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1. Introduction

LET R_{n+m} be the group of rotations of S^{n+m-1} and let $R_n \subset R_{n+m}$ be the subgroup of rotations in the plane of the first n axes. In this note a homomorphism

$$P_m\colon \pi_r(R_{n+m}, R_n) \to \pi_{r+n-1}(S^n)$$

is defined and also, if $r \leqslant 2n-2$, a homomorphism

$$H_m\colon \pi_{r+n+m}(S^{n+m}) \to \pi_r(R_{n+m}, R_n).$$

I shall prove

THEOREM (1.1). *The following sequence is exact:*

$$\pi_{3n-2}(S^n) \to \ldots \to \pi_{r+n}(S^n) \xrightarrow{E^m} \pi_{r+n+m}(S^{n+m}) \xrightarrow{H_m} \xrightarrow{H_m} \pi_r(R_{n+m}, R_n) \xrightarrow{P_m} \pi_{r+n-1}(S^n) \to \ldots.$$

In (1.1), E^m is the resultant of m successive Freudenthal suspension operators (m-fold suspension). A number of applications will be found in § 5 below.

When $m = 1$, the above sequence is a variant of the exact sequence which is the subject of G. W. Whitehead's paper 'On the Freudenthal theorems' [(**6**), Theorem 1]. To make this statement precise, let $\pi_r(R_{n+1}, R_n)$ be identified with $\pi_r(S^n)$ by the isomorphism induced by the fibre-mapping $R_{n+1} \to S^n$. Let $P\colon \pi_r(S^n) \to \pi_{r+n-1}(S^n)$ be the Whitehead product-homomorphism given by

$$P\alpha = [\alpha, \iota_n] \quad (\alpha \in \pi_r(S^n)),$$

where $\iota_n \in \pi_n(S^n)$ is represented by the identity map. If $r \leqslant 2n-2$, let H be the generalized Hopf invariant (**5**) and let E^{n+1} be the $(n+1)$-fold suspension isomorphism

$$\pi_{r+n+1}(S^{n+1}) \xrightarrow{H} \pi_{r+n+1}(S^{2n+1}) \xleftarrow{E^{n+1}} \pi_r(S^n).$$

With the definitions of P_m, H_m given below we shall have

$$\left.\begin{array}{ll} (a) & P_1 = -P \\ (b) & H_1 = -E^{-n-1} \circ H \end{array}\right\}. \tag{1.2}$$

It follows from (5.3) of (**6**) that Whitehead's exact sequence may be written in the form

$$\pi_{3n-2}(S^n) \to \dots \to \pi_{r+n}(S^n) \xrightarrow{E} \xrightarrow{E} \pi_{r+n+1}(S^{n+1}) \xrightarrow{E^{-n-1} \circ H} \pi_r(S^n) \xrightarrow{P} \pi_{r+n-1}(S^n) \to \dots \quad (1.3)$$

and, because of (1.2), the sequence of (1.1) is obtained from (1.3) by changing the signs of $E^{-n-1} \circ H$, P.

2. The function spaces

On the sphere S^{n+1} let y_0 be the north pole, S^n the equator, and $x_0 \in S^n$ a point on the equator. With the compact-open topology let F_{n+1} be the function space of maps $(S^{n+1}, y_0) \to (S^{n+1}, y_0)$ of degree 1. Let $G_n \subset F_{n+1}$ be the subset consisting of suspensions of maps $S^n \to S^n$, and let $F_n \subset G_n$ be the subset which also carry $x_0 \to x_0$. It is convenient to identify G_n with the space of maps $S^n \to S^n$ of degree 1. Let

$$p\colon (G_n, F_n) \to (S^n, x_0)$$

be the fibre mapping defined by $pf = fx_0$ $(f \in G_n)$. I assert†

THEOREM (2.1). *If $r \leqslant 2n-2$, then*‡

$$\pi_r(G_n, F_n) \approx \pi_r(F_{n+1}, F_n)$$

under the injection.

In the following diagram, where $r \leqslant 2n-2$, the lower sequence is (1.3).

$$\begin{array}{ccccccccc}
\dots \to \pi_r(F_n) & \xrightarrow{i'} & \pi_r(F_{n+1}) & \xrightarrow{h} & \pi_r(G_n, F_n) & \xrightarrow{\partial} & \pi_{r-1}(F_n) \to \dots \\
\downarrow \zeta & & \downarrow \eta & & \downarrow p_* & & \downarrow \zeta \\
\dots \to \pi_{r+n}(S^n) & \xrightarrow{E} & \pi_{r+n+1}(S^{n+1}) & \xrightarrow{E^{-n-1} \circ H} & \pi_r(S^n) & \xrightarrow{P} & \pi_{r+n-1}(S^n) \to \dots
\end{array}$$

p_* is the isomorphism induced by p, and ζ, η are Hurewicz isomorphisms. Finally i' is the injection, ∂ is the boundary operator, and

$$h = p_*^{-1} \circ E^{-n-1} \circ H \circ \eta.$$

By (3.2), (3.10) of (**4**), as corrected in (**8**), we have

$$P \circ p_* = -\zeta \circ \partial, \qquad E \circ \zeta = -\eta \circ i'.$$

† I am grateful to Professor J. H. C. Whitehead, who kindly read the manuscript, for improving the proof of this theorem.

‡ In the homotopy groups of these function spaces the base-point is the identity map.

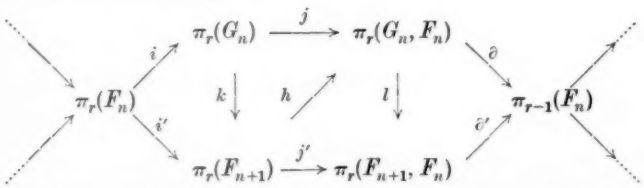
Because the lower sequence is exact starting with†

$$E\colon \pi_{3n-2}(S^n) \to \pi_{3n-1}(S^{n+1}),$$

it follows that the upper sequence is exact starting with

$$i'\colon \pi_{2n-2}(F_n) \to \pi_{2n-2}(F_{n+1}).$$

Consider next the diagram



which indicates the injection of the homotopy sequence of the pair (G_n, F_n) into that of (F_{n+1}, F_n), together with h if $r \leqslant 2n-2$. Besides the naturality relations $k \circ i = i'$, $\partial' \circ l = \partial$, and $l \circ j = j' \circ k$, we have $h \circ k = -j$ from (5.1) of (**5**) [cf. (**8**)]. On comparing the sequence i', h, ∂,..., which is exact, starting with i': $\pi_{2n-2}(F_n) \to \pi_{2n-2}(F_{n+1})$, against the exact sequence i, j, ∂,..., it follows from the five lemma‡ and (7.4) in (**1**) that k: $\pi_r(G_n) \approx \pi_r(F_{n+1})$ if $r < 2n-2$ and that

$$k\pi_{2n-2}(G_n) = \pi_{2n-2}(F_{n+1}).$$

However $$l \circ h \circ k = -l \circ j = -j' \circ k,$$

and, since k is onto if $r \leqslant 2n-2$, we conclude that $l \circ h = -j'$. Hence it follows from the five lemma, applied to the sequences i', h, ∂,... and i', j', ∂',..., that l: $\pi_r(G_n, F_n) \approx \pi_r(F_{n+1}, F_n)$ if $r \leqslant 2n-2$. This proves (2.1) and incidentally

COROLLARY (2.2). *The pair* (F_{n+1}, G_n) *is* $(2n-2)$*-connected.*

It would be interesting to determine the first non-vanishing homotopy group $\pi_r(F_{n+1}, G_n)$.

3. Proof of (1.1)

Let $R_{n+1} \subset G_n$ be the group of rotations of S^n and let $R_n = F_n \cap R_{n+1}$ be the isotropic subgroup. Consider the injections

$$\pi_r(R_{n+1}, R_n) \xrightarrow{l'} \pi_r(G_n, F_n) \xrightarrow{l} \pi_r(F_{n+1}, F_n).$$

† That is to say, the sequence is exact at $\pi_{3n-1}(S^{n+1})$ and subsequently.

‡ An exact sequence remains exact if some of the homomorphisms are reversed in sign. It follows that for the purposes of the five lemma etc., it is immaterial if some of the 'commutativity' relations are not commutative but anti-commutative.

Since p and $p|(R_{n+1}, R_n)$ are fibre mappings, we have

$$p_*\colon \pi_r(G_n, F_n) \approx \pi_r(S^n),$$

$$p'_*\colon \pi_r(R_{n+1}, R_n) \approx \pi_r(S^n),$$

where $p'_* = p_* \circ l'$. Hence l' is an isomorphism onto and (2.1) can be restated as

LEMMA (3.1). *If $r \leqslant 2n-2$, then*

$$\pi_r(R_{n+1}, R_n) \approx \pi_r(F_{n+1}, F_n)$$

under the injection.

We define F_{n+m} in analogy with F_{n+1} as the space of maps $S^{n+m} \to S^{n+m}$ of degree 1 which leave the north pole fixed. We embed successively $F_n \subset \ldots \subset F_q \subset \ldots \subset F_{n+m}$, so that $R_q = F_q \cap R_{n+m}$ if $n \leqslant q \leqslant n+m$. Applying the five lemma to the injection of the homotopy sequence of the triple $(R_{n+m}, R_{n+m-1}, R_n)$ into that of $(F_{n+m}, F_{n+m-1}, F_n)$ we obtain from (3.1) an inductive proof of

THEOREM (3.2). *If $r \leqslant 2n-2$, then*

$$\phi_m\colon \pi_r(R_{n+m}, R_n) \approx \pi_r(F_{n+m}, F_n),$$

where ϕ_m is the injection.

In passing I make the following deduction† from (3.2). Consider the injection of the homotopy sequence of the pair (R_{n+m}, R_n) into that of (F_{n+m}, F_n), as shown in the following diagram:

$$\begin{array}{ccccccc}
\pi_{r+1}(R_{n+m}, R_n) & \xrightarrow{\partial_R} & \pi_r(R_n) & \xrightarrow{i_R} & \pi_r(R_{n+m}) & \xrightarrow{j_R} & \pi_r(R_{n+m}, R_n) \\
\downarrow{\scriptstyle \phi_m} & & \downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle i_{n+m}} & & \downarrow{\scriptstyle \phi_m} \\
\pi_{r+1}(F_{n+m}, F_n) & \xrightarrow{\partial_F} & \pi_r(F_n) & \xrightarrow{i_F} & \pi_r(F_{n+m}) & \xrightarrow{j_F} & \pi_r(F_{n+m}, F_n)
\end{array}$$

When $r \leqslant 2n-3$, both ϕ_m shown above are isomorphisms onto by (3.2). Hence it follows from (7.1), (7.2) of (**1**) that

$$i_{n+m}^{-1}(0) = i_R(i_n^{-1}(0)),$$

$$i_n \pi_r(R_n) = i_F^{-1}(i_{n+m}\pi_r(R_{n+m})).$$

Let $J_q\colon \pi_r(R_q) \to \pi_{r+q}(S^q)$ $(q = n, n+m)$ be the composition of i_q and the Hurewicz isomorphism. Then we obtain

† This can also be proved directly from (1.3).

THEOREM (3.3). *Let $r \leqslant 2n-3$. Then we have*

$$(a) \quad J_{n+m}^{-1}(0) = i_R(J_n^{-1}(0)) \quad \text{in } \pi_r(R_{n+m}),$$

$$(b) \quad J_n \pi_r(R_n) = E^{-m}(J_{n+m}\pi_r(R_{n+m})).$$

Retaining the above notation I conclude this section by proving (1.1). Let $P_m = J_n \circ \partial_R$,

$$\pi_r(R_{n+m}, R_n) \xrightarrow[\partial_R]{} \pi_{r-1}(R_n) \xrightarrow[J_n]{} \pi_{r+n-1}(S^n).$$

An equivalent definition is $P_m = \zeta \circ \partial_F \circ \phi_m$,

$$\pi_r(R_{n+m}, R_n) \xrightarrow[\phi_m]{} \pi_r(F_{n+m}, F_n) \xrightarrow[\partial_F]{} \pi_{r-1}(F_n) \xrightarrow[\zeta]{} \pi_{r+n-1}(S^n),$$

ζ being the Hurewicz isomorphism, because $J_n = \zeta \circ i_n$ and

$$i_n \circ \partial_R = \partial_F \circ \phi_m.$$

Provided that $r \leqslant 2n-2$, (3.2) enables us to define

$$H_m = \phi_m^{-1} \circ j_F \circ \eta^{-1},$$

$$\pi_{r+n+m}(S^{n+m}) \xleftarrow[\eta]{} \pi_r(F_{n+m}) \xrightarrow[j_F]{} \pi_r(F_{n+m}, F_n) \xleftarrow[\phi_m]{} \pi_r(R_{n+m}, R_n),$$

where η is the Hurewicz isomorphism. P_m, H_m are the homomorphisms referred to in § 1.

It follows from (3.10) of (**4**) [cf. (**8**)] that $\eta \circ i_F = (-1)^m E^m \circ \zeta$,

$$\begin{array}{ccc} \pi_r(F_n) & \xrightarrow{i_F} & \pi_r(F_{n+m}) \\ \zeta \downarrow & & \downarrow \eta \\ \pi_{r+n}(S^n) & \xrightarrow{E^m} & \pi_{r+n+m}(S^{n+m}). \end{array} \tag{3.4}$$

In the homotopy sequence of the pair (F_{n+m}, F_n) we identify $\pi_r(F_n)$ with $\pi_{r+n}(S^n)$ by ζ, $\pi_r(F_{n+m})$ with $\pi_{r+n+m}(S^{n+m})$ by η, and, if $r \leqslant 2n-2$, $\pi_r(F_{n+m}, F_n)$ with $\pi_r(R_{n+m}, R_n)$ by ϕ_m^{-1}. The result is the exact sequence:

$$\pi_{3n-2}(S^n) \to \ldots \to \pi_{r+n}(S^n) \xrightarrow{(-1)^m E^m} \pi_{r+n+m}(S^{n+m}) \xrightarrow{H_m}$$

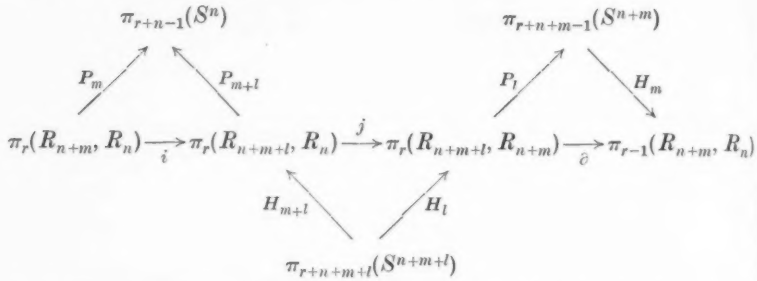
$$\xrightarrow{H_m} \pi_r(R_{n+m}, R_n) \xrightarrow{P_m} \pi_{r+n-1}(S^n) \to \ldots .$$

Replacing $(-1)^m E^m$ by E^m, the exactness of the sequence in (1.1) is established.

4. Proof of (1.2) and other relations

There are a number of relations involving the homomorphisms P_m, H_m. Let $l \geqslant 0$ be an integer. In the following diagram i, j, ∂ are now the

injections and boundary operator in the homotopy sequence of the triple $(R_{n+m+l}, R_{n+m}, R_n)$.



A short naturality argument shows that the triangles in the above diagram are commutative, as asserted in

THEOREM (4.1). *We have the relations*

(a) $P_m = P_{m+l} \circ i$;

(b) $H_l = j \circ H_{m+l}$, *if* $r \leqslant 2n-2$;

(c) $\partial = H_m \circ P_l$, *if* $r \leqslant 2n-1$.

Clearly the composition

$$\pi_r(R_{n+m}) \xrightarrow[J_{n+m}]{} \pi_{r+n+m}(S^{n+m}) \xrightarrow[H_m]{} \pi_r(R_{n+m}, R_n),$$

when defined, is equal to the injection j_R. Also, from (3.4), it follows that $P_l \circ j = (-1)^m E^m \circ P_{m+l}$,

$$\begin{array}{ccc} \pi_r(R_{n+m+l}, R_n) & \xrightarrow{j} & \pi_r(R_{n+m+l}, R_{n+m}) \\ \Big\downarrow{\scriptstyle P_{m+l}} & & \Big\downarrow{\scriptstyle P_l} \\ \pi_{r+n-1}(S^n) & \xrightarrow{E^m} & \pi_{r+n+m-1}(S^{n+m}) \end{array}$$

and, if $r \leqslant 2n-2$, that $i \circ H_m = (-1)^l H_{m+l} \circ E^l$,

$$\begin{array}{ccc} \pi_{r+n+m}(S^{n+m}) & \xrightarrow{E^l} & \pi_{r+n+m+l}(S^{n+m+l}) \\ \Big\downarrow{\scriptstyle H_m} & & \Big\downarrow{\scriptstyle H_{m+l}} \\ \pi_r(R_{n+m}, R_n) & \xrightarrow{i} & \pi_r(R_{n+m+l}, R_n). \end{array}$$

These results are collected into

THEOREM (4.2). *We have the relations*

$$(a) \quad j_R = H_m \circ J_{n+m}, \quad \text{if } r \leqslant 2n-2;$$

$$(b) \quad P_l \circ j = (-1)^m E^m \circ P_{m+l};$$

$$(c) \quad i \circ H_m = (-1)^l H_{m+l} \circ E^l, \quad \text{if } r \leqslant 2n-2.$$

Let $l = 1$. We regard S^{n+m} as the factor space R_{n+m+1}/R_{n+m} and we identify $\pi_r(R_{n+m+1}, R_{n+m})$ with $\pi_r(S^{n+m})$ by the isomorphism induced by the fibre mapping. Then the injection $j = t_* \circ s_*$,

$$\pi_r(R_{n+m+1}, R_n) \xrightarrow[s_*]{} \pi_r(R_{n+m+1}/R_n) \xrightarrow[t_*]{} \pi_r(S^{n+m}),$$

where s_*, t_* are induced by the natural maps,

$$R_{n+m+1} \xrightarrow{s} R_{n+m+1}/R_n \xrightarrow{t} S^{n+m}.$$

Suppose that S^{n+m} admits an m-field. Then t has a right inverse map and so t_* is onto; s_* is an isomorphism since s is a fibre mapping; hence j is onto. By exactness it follows that i is an isomorphism into.† Hence we deduce from (4.2b), (1.2a) and (4.2c)

THEOREM (4.3). *Suppose that S^{n+m} admits an m-field. Then*

$$P\pi_r(S^{n+m}) = E^m \circ P_{m+1}\pi_r(R_{n+m+1}, R_n),$$

and, if $r \leqslant 2n-2$,

$$H_m^{-1}(0) = E^{-1}(H_{m+1}^{-1}(0)) \quad \text{in } \pi_{r+n+m}(S^{n+m}).$$

It remains to identify P_1, H_1 as in (1.2). (1.2a) follows at once from (3.2) of (**4**) [cf. (**8**)]. To prove (1.2b) recall the relation $l \circ h = -j'$ occurring in the proof of (2.1). Since the homomorphism j' in § 2 is the same as j_F in § 4, we have $H_1 = \phi_1^{-1} \circ j' \circ \eta^{-1}$. Therefore

$$\begin{aligned} p'_* \circ H_1 = p_* \circ l^{-1} \circ j' \circ \eta^{-1} &= -p_* \circ h \circ \eta^{-1} \\ &= -E^{-n-1} \circ H, \end{aligned}$$

by the definition of h. This proves (1.2b).

5. Applications

By the 'r-stem' is meant the sequence of homotopy groups $\pi_{r+q}(S^q)$ ($q = 1, 2, \ldots$). The suspension maps the groups for which $q \geqslant r+2$ isomorphically onto; the limit group is called the *stable* group of the

† In fact j has a right inverse and i has a left inverse.

r-stem. There follows from (1.1)

THEOREM (5.1). *Let $r \geqslant 0$. Suppose that the stable group of the r-stem is trivial. Then we have, for $m = r-n+2 \geqslant 0$,*

(*a*) $P_m \pi_{r+1}(R_{r+2}, R_n) = \pi_{r+n}(S^n)$, *if* $r \leqslant 2n-3$;

(*b*) P_m: $\pi_r(R_{r+2}, R_n) \to \pi_{r+n-1}(S^n)$ *is an isomorphism into, if* $r \leqslant 2n-2$.

Similarly there follows

THEOREM (5.2). *Let $r \geqslant 0$. Suppose that the stable groups of the r-stem and of the $(r+1)$-stem are trivial. Then, for any n such that $n \leqslant r+3 \leqslant 2n$, we have*

$$P_{r-n+3}\colon \pi_{r+1}(R_{r+3}, R_n) \approx \pi_{r+n}(S^n).$$

According to (**2**), (**3**) the stable groups of the 4-stem and of the 5-stem are trivial, so that the hypotheses of (5.1), (5.2) are not unrealizable.

Another application of (1.1) is

THEOREM (5.3). *Let m, n, r be integers such that*

$$E^m\colon \pi_{r+n}(S^n) \approx \pi_{r+n+m}(S^{n+m}).$$

Then, if $q \leqslant m$ and $r \leqslant 2(n+q)-3$, there is a direct-sum decomposition

$$\pi_{r+n+q}(S^{n+q}) = E^q \pi_{r+n}(S^n) + P_{m-q}\pi_{r+1}(R_{n+m}, R_{n+q})$$

in which E^q is an isomorphism into.

For, because E^m is an isomorphism, it follows that

$$E^q\colon \pi_{r+n}(S^n) \to \pi_{r+n+q}(S^{n+q})$$

is an isomorphism into and that $E^{m-q}\colon \pi_{r+n+q}(S^{n+q}) \to \pi_{r+n+m}(S^{n+m})$ has a right inverse $E^q \circ E^{-m}$. If we use this inverse in (1.1) with $n+q$ instead of n and $m-q$ instead of m, (5.3) follows at once. The same kind of argument used with the relation (4.2*a*) proves

THEOREM (5.4). *Let m, n, r be integers ($r \leqslant 2n-2$) such that*

$$\pi_r(R_{n+m}) \approx \pi_r(R_{n+m}, R_n)$$

under the injection. Then there is a direct-sum decomposition

$$\pi_{r+n+m}(S^{n+m}) = E^m \pi_{r+n}(S^n) + J_{n+m}\pi_r(R_{n+m}),$$

in which J_{n+m} is an isomorphism into.

A third condition leading to a direct sum, proved on the same lines from (4.1 c), is the subject of

THEOREM (5.5). *Let l, m, n, r be integers ($r \leqslant 2n-2$) such that the boundary homomorphism*

$$\partial\colon \pi_{r+1}(R_{n+m+l}, R_{n+m}) \approx \pi_r(R_{n+m}, R_n).$$

Then there is a direct-sum decomposition

$$\pi_{r+n+m}(S^{n+m}) = E^m\pi_{r+n}(S^n)+P_l\pi_{r+1}(R_{n+m+l}, R_{n+m})$$

in which P_l is an isomorphism into.

We notice

COROLLARY (5.6). *Let m, n, r be integers ($r \leqslant 2n-3$) such that $\pi_p(R_{n+m}, R_n) = 0$ if $p = r$, $r+1$. Then, if $q \leqslant m$, there is a direct-sum decomposition*

$$\pi_{r+n+q}(S^{n+q}) = E^q\pi_{r+n}(S^n)+P_{m-q}\pi_{r+1}(R_{n+m}, R_{n+q})$$

in which E^q, P_{m-q} are isomorphisms into.

For it follows from the homotopy sequence of the triple

$$(R_{n+m}, R_{n+q}, R_n)$$

that $$\partial\colon \pi_{r+1}(R_{n+m}, R_{n+q}) \approx \pi_r(R_{n+q}, R_n),$$

and directly from (1.1) that E^q is an isomorphism into.

We have that $\pi_r(R_{n+m}, R_n) \approx \pi_r(V_{n+m,m})$, where $V_{n+m,m}$ is the Stiefel manifold of m-frames at the origin in Euclidean $(n+m)$-space. Moreover by Theorem 3 of (7), if $r \leqslant 2n-1$, we have

$$\pi_r(V_{n+m,m}) \approx \pi_r(P^l_{n-1}) \quad (l = \min(r+1, n+m-1)),$$

where P^l_{n-1} is the space obtained from real projective l-space by shrinking an $(n-1)$-dimensional linear subspace to a point. In (1.1), $r \leqslant 2n-2$, so that this last isomorphism is applicable. Extensive computations of these homotopy groups by G. F. Paechter are to be published shortly.† According to these, for any $m \geqslant 0$, the injection $\pi_7(R_5) \to \pi_7(R_{m+5})$ is an isomorphism into and $\pi_8(R_5) = 0$. Hence $\pi_8(R_{m+5}) \approx \pi_8(R_{m+5}, R_5)$ under the injection, so that (5.4) gives

COROLLARY (5.7). *Let $m \geqslant 0$. Then there is a direct-sum decomposition*

$$\pi_{m+13}(S^{m+5}) = E^m\pi_{13}(S^5)+J_{m+5}\pi_8(R_{m+5})$$

in which J_{m+5} is an isomorphism into.

† I am grateful to Dr. Paechter for allowing me to quote results from his thesis.

Also according to Paechter's computations, if $n \equiv 5 \pmod 8$, we have $\pi_p(R_{n+7}, R_n) = 0$ for $p = n+4, n+5$. Assuming this, we deduce from (5.6)

COROLLARY (5.8). *Let* $n \equiv 5 \pmod 8$ $(n \geqslant 13)$. *Then, if* $q \leqslant 7$, *there is a direct-sum decomposition*

$$\pi_{2n+q+4}(S^{n+q}) = E^q\pi_{2n+4}(S^n) + P_{7-q}\pi_{n+5}(R_{n+7}, R_{n+q})$$

in which E^q, P_{7-q} *are isomorphisms into.*

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HYDRODYNAMIC FORCES ON OBSTACLES DUE TO LINE SOURCES

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1. Introduction

It has been shown by Milne-Thomson (**1**) that the force $\mathbf{F}$ and the couple $\mathbf{G}_0$ about the origin $\mathbf{r} = \mathbf{0}$ experienced by an obstacle immersed in a fluid moving steadily and irrotationally in the absence of body forces can be written as

$$\mathbf{F}/\rho = \sum_{i=1}^{k} \int_{(S_i)} \mathbf{\Gamma}\, dS + \int_{(E)} \mathbf{\Gamma}\, dS, \qquad \mathbf{G}_0/\rho = \sum_{i=1}^{k} \int_{(S_i)} \mathbf{r}\wedge\mathbf{\Gamma}\, dS + \int_{(E)} \mathbf{r}\wedge\mathbf{\Gamma}\, dS \tag{1}$$

where $\mathbf{\Gamma} = \mathbf{q}(\mathbf{n}.\mathbf{q}) - \frac{1}{2}\mathbf{n}q^2$, the S_i ($i = 1, 2, \ldots, k$) being k surfaces tightly enclosing the k singularities of flow such as sources and doublets, etc., whilst E is the fluid envelope and $\mathbf{n}$ the unit normal at any point of S_i or E drawn into the fluid. It is also proved in (**1**) that, when the fluid velocity $\mathbf{q}$ behaves like $O(r^{-2})$ when r is large, the contribution of the envelope E to both $\mathbf{F}$ and $\mathbf{G}_0$ tends to zero when $r \to \infty$. However, this sufficiency condition is of no use in streaming-motion problems, so that in general we have to assume a slightly more complicated form for the velocity at large r. Consider

$$\mathbf{q} = \mathbf{u} + \mathbf{v}r^{-2} + \mathbf{w}r^{-3} + O(r^{-4}),$$

where $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are at the most $O(1)$ as $r \to \infty$. Then

$$\int_{(E)} \mathbf{\Gamma}\, dS = \int_{(E)} \{\mathbf{u}(\mathbf{n}.\mathbf{u}) - \tfrac{1}{2}\mathbf{n}u^2\}\, dS + \int_{(E)} \{\mathbf{u}(\mathbf{n}.\mathbf{v}) + \mathbf{v}(\mathbf{n}.\mathbf{u}) - \mathbf{n}(\mathbf{u}.\mathbf{v})\} r^{-2}\, dS + O(r^{-1}).$$

A sufficient condition that this integral of $\mathbf{\Gamma}$ over E should tend to a finite value as the envelope increases indefinitely is that $\mathbf{u}$ should be a constant vector, i.e. uniform streaming motion at infinity. Furthermore, the contribution of E is zero as $r \to \infty$ if in addition $\mathbf{v}$ is also a constant vector. However, in the case of a line distribution of sources or sinks it is found that $\mathbf{v} = A\hat{\mathbf{r}}$ ($\hat{\mathbf{r}}$ being the unit vector in the direction of $\mathbf{r}$ and

A a constant scalar). If E is taken as a spherical surface centred at the origin, then $\mathbf{n} = -\hat{\mathbf{r}}$, so that

$$\int_{(E)} \mathbf{\Gamma}\, dS = -4\pi A\mathbf{u} + O(r^{-1}).$$

In the case of the couple I have to consider

$$\begin{aligned}\int_{(E)} \mathbf{r}\wedge\mathbf{\Gamma}\, dS = \int_{(E)} \mathbf{r}\wedge\{\mathbf{u}(\mathbf{n}.\mathbf{u}) - \tfrac{1}{2}\mathbf{n}u^2\}\, dS + \\ + \int_{(E)} \mathbf{r}\wedge\{\mathbf{u}(\mathbf{n}.\mathbf{v}) + \mathbf{v}(\mathbf{n}.\mathbf{u}) - \mathbf{n}(\mathbf{u}.\mathbf{v})\} r^{-2}\, dS + \\ + \int_{(E)} \mathbf{r}\wedge\{\mathbf{w}(\mathbf{n}.\mathbf{u}) + \mathbf{u}(\mathbf{n}.\mathbf{w}) - \mathbf{n}(\mathbf{u}.\mathbf{w})\} r^{-3}\, dS + \\ + O(r^{-1}).\end{aligned}$$

Again, taking E as a large spherical surface centred at $\mathbf{r} = 0$, we have $\mathbf{n} = -\hat{\mathbf{r}}$ and, since

$$\int_{(E)} \mathbf{r}\wedge\{\mathbf{u}(\mathbf{r}.\mathbf{v}) + \mathbf{v}(\mathbf{r}.\mathbf{u})\}\, dS = 0$$

when $\mathbf{u}$ and $\mathbf{v}$ are constant vectors, it follows that the contribution of the envelope to $\mathbf{G}_0$ will be finite when $\mathbf{u}$ and $\mathbf{v}$ are constant vectors. On the other hand, if $\mathbf{u}$ is a constant and $\mathbf{v} = A\hat{\mathbf{r}}$, where A is a constant scalar, then the contribution is still finite.

The hydrodynamic singularities considered in (**1**) are the point source and doublet respectively. If m is the strength of a source at $\mathbf{r}_i$ and $\mathbf{Q}_i$ the fluid velocity at this point due to all effects excepting the source there, the force due to this source is $4\pi m\mathbf{Q}_i$. Similarly, in the case of a doublet of strength and direction represented by $\boldsymbol{\mu}$, the force is $4\pi(\boldsymbol{\mu}.\nabla)\mathbf{Q}_i$. In this paper I derive expressions for the force and couple due to a straight-line distribution of sources of variable density, the results being given as integrals of the fluid velocity $\mathbf{Q}$ due to all other effects excepting the line source. The effect of the envelope E has not been included in these results since it will vary from one problem to another. However, the procedure in any particular case has been described in the preceding paragraph.

The result is utilized at the end of the paper to solve the mutual-interference problem of n airship forms placed in a line parallel to a uniform stream. The method is an extremely useful one in dealing with interference problems of this type. I hope eventually to publish corresponding results for other source configurations and also the horseshoe vortex, which will possibly open up the way to the solutions of certain unsolved problems.

2. The velocity potential

Let the straight-line distribution of sources lie along the **i**-axis from $\mathbf{r} = 0$ to $\mathbf{r} = a\mathbf{i}$, the density at $\sigma\mathbf{i}$ being $f(\sigma)$. The velocity potential of fluid motion due to this system is the integral

$$\int_0^a \frac{f(\sigma)\, d\sigma}{|\mathbf{r}-\sigma\mathbf{i}|}. \tag{2}$$

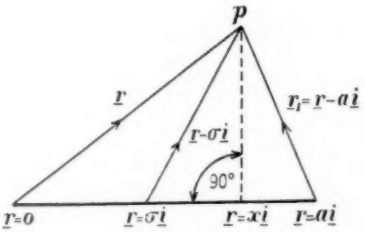

FIG. 1.

If $f(\sigma) = 1$, this integral, denoted by I_0 is simply

$$I_0 = \log_e\left(\frac{r+r_1+a}{r+r_1-a}\right), \tag{3}$$

where $\mathbf{r}_1 = \mathbf{r}-a\mathbf{i}$ and $|\mathbf{r}_1| = r_1$. Again, if $f(\sigma) = \sigma^n$ (n a positive integer), then denoting the integral in (2) by I_n we find that the following recurrence relationship holds

$$\left.\begin{aligned} nI_n &= (2n-1)xI_{n-1}-(n-1)r^2I_{n-2}+a^{n-1}r_1 \quad (n \geqslant 2)\\ I_1 &= xI_0+r_1-r \end{aligned}\right\}, \tag{4}$$

so that the form of I_n can be written as

$$I_n = P_n(x,r)I_0+S_n(x,r)$$

where $P_n(x,r)$ is polynomial in x and r^2, and both P_n and S_n are certainly defined and finite in the immediate neighbourhood of the line between $\mathbf{r} = 0$ and $\mathbf{r} = a\mathbf{i}$. If on the other hand we take $f(x)$ as a function having a Maclaurin expansion $\sum_0^\infty a_n x^n$ with radius of convergence R, then, since the series is uniformly convergent in the open interval $(-R, R)$, we may, provided that $a < R$, integrate term by term having divided throughout by $|\mathbf{r}-\sigma\mathbf{i}|$, i.e.

$$\int_0^a \frac{f(\sigma)\, d\sigma}{|\mathbf{r}-\sigma\mathbf{i}|} = \sum_{n=0}^\infty a_n I_n \quad (a < R).$$

Denoting by Φ the velocity potential of the fluid motion due to all other effects excluding the line distribution we can write the final velocity potential as

$$\left.\begin{aligned} \phi &= \Phi+\phi_0(x,r)\log_e\left(\frac{r+r_1+a}{r+r_1-a}\right)+\phi_1(x,r) \\ \text{where}\qquad \phi_0 &= \sum_0^\infty a_n P_n(x,r), \qquad \phi_1 = \sum_0^\infty a_n S_n(x,r) \end{aligned}\right\}, \tag{5}$$

so that ϕ_0 and ϕ_1 are defined and finite in the immediate neighbourhood of the given line singularity and ϕ_0 is a function of r through r^2.

If $f(x)$ be polynomial in x, relations (5) hold with the added advantage that no limit need be placed on the value of a.

3. The fluid velocity

The fluid velocity $\mathbf{q}$ can be written down from (5). In fact, if $\mathbf{Q}$ is the fluid velocity due to the velocity potential Φ,

$$\mathbf{Q} = -\text{grad}\,\Phi, \tag{6}$$

so that

$$\left.\begin{aligned} \mathbf{q} &= -\text{grad}\,\phi \\ &= \mathbf{Q}-\mathbf{q}_0\log_e\left(\frac{r+r_1-a}{r+r_1+a}\right)+(\hat{\mathbf{r}}+\hat{\mathbf{r}}_1)\left(\frac{1}{r+r_1-a}-\frac{1}{r+r_1+a}\right)\phi_0+\mathbf{q}_1 \\ \text{where}\qquad \mathbf{q}_0 &= -\text{grad}\,\phi_0, \qquad \mathbf{q}_1 = -\text{grad}\,\phi_1 \end{aligned}\right\}. \tag{7}$$

4. The surface of integration

I shall take as the tight surface enclosing the line distribution the right circular cylindrical surface having the source, i.e. $\mathbf{i}$, as axis and of length equal to that of the distribution. The radius of the circular

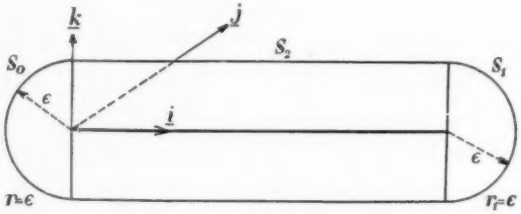

FIG. 2.

section is taken as ϵ (small enough to exclude all other possible hydrodynamic singularities in the vicinity) and the surface is made continuous by the introduction of hemispherical caps S_0 and S_1 whose equations are $r = |\mathbf{r}| = \epsilon$ and $r_1 = |\mathbf{r}-a\mathbf{i}| = \epsilon$ respectively. For future work I shall refer to the cylindrical part of the surface as S_2.

5. The integration of Γ over S_0 and S_1

First, I examine the expression for the velocity $\mathbf{q}$ on S_0 and hence obtain an expression for $\mathbf{\Gamma}$. On this surface $\mathbf{r} = \epsilon\mathbf{n}$, where $\mathbf{n}$ is the unit normal $\mathbf{i}\cos\theta + \boldsymbol{\omega}\sin\theta$, $\boldsymbol{\omega}$ being the variable unit vector perpendicular to $\mathbf{i}$ and θ the angle which $\mathbf{r}$ makes with $\mathbf{i}$. Now on S_0 it can be shown that

$$\log\left(\frac{r+r_1-a}{r+r_1+a}\right) = \log\epsilon + O(1), \tag{8.1}$$

$$(\hat{\mathbf{r}}+\hat{\mathbf{r}}_1)\left(\frac{1}{r+r_1-a}-\frac{1}{r+r_1+a}\right) = \frac{\mathbf{n}-\mathbf{i}}{\epsilon(1-\cos\theta)}+\frac{\mathbf{i}}{a}+O(\epsilon), \tag{8.2}$$

$$\phi_0(x,r) = \phi_0(0,0)+O(\epsilon), \tag{8.3}$$

$$\mathbf{q}_0 = O(1), \qquad \phi_1 = O(1), \qquad \mathbf{q}_1 = O(1). \tag{8.4}$$

Hence, from equations (7) and (8), it appears that, on S_0, I can write $\mathbf{q}$ as

$$\mathbf{S}_0\log\epsilon + \mathbf{S}_1/\epsilon + O(1),$$

where $\mathbf{S}_0 = O(1)$, $\mathbf{S}_1 = (\mathbf{n}-\mathbf{i})\phi_0(0,0)/(1-\cos\theta)$.

Now, it is quite straightforward to show that

$$\left.\begin{aligned} &\mathbf{\Gamma} = \mathbf{q}(\mathbf{n}.\mathbf{q})-\tfrac{1}{2}\mathbf{n}q^2 = \mathbf{\Gamma}_2\epsilon^{-2}+O(\epsilon^{-1}\log\epsilon)\\ \text{where}\quad &\mathbf{\Gamma}_2 = \mathbf{S}_1(\mathbf{n}.\mathbf{S}_1)-\tfrac{1}{2}\mathbf{n}S_1^2 = -\gamma_0^2\mathbf{i}/(1-\cos\theta)\\ \text{and}\quad &\gamma_0 = \phi_0(0,0)\end{aligned}\right\}. \tag{9}$$

By using the same argument a similar expression can be obtained for $\mathbf{\Gamma}$ on S_1. For on this surface $\mathbf{r} = a\mathbf{i}+\epsilon\mathbf{n}$, where $\mathbf{n}$ is the unit outward normal $i\cos\theta+\boldsymbol{\omega}\sin\theta$ as before, so that

$$\log\left(\frac{r+r_1-a}{r+r_1+a}\right) = \log\epsilon+O(1), \tag{10.1}$$

$$(\hat{\mathbf{r}}+\hat{\mathbf{r}}_1)\left(\frac{1}{r+r_1-a}-\frac{1}{r+r_1+a}\right) = \frac{\mathbf{n}+\mathbf{i}}{\epsilon(1+\cos\theta)}+O(1), \tag{10.2}$$

$$\phi_0(x,r) = \phi_0(a,a)+O(\epsilon), \tag{10.3}$$

and, since relations (8.4) hold on S_1, then on this surface

$$\mathbf{q} = \mathbf{S}_0'\log\epsilon+\mathbf{S}_1'/\epsilon+O(1),$$

where $\mathbf{S}_0' = O(1)$, $\mathbf{S}_1' = (\mathbf{n}-\mathbf{i})\phi_0(a,a)/(1+\cos\theta)$; so that, finally,

$$\left.\begin{aligned} &\mathbf{\Gamma} = \mathbf{\Gamma}_2'\epsilon^{-2}+O(\epsilon^{-1}\log\epsilon),\\ \text{where}\quad &\mathbf{\Gamma}_2' = \mathbf{S}_1'(\mathbf{n}.\mathbf{S}_1')-\tfrac{1}{2}\mathbf{n}S_1'^2 = \gamma_a^2\mathbf{i}/(1+\cos\theta)\\ \text{and}\quad &\gamma_a = \phi_0(a,a)\end{aligned}\right\}. \tag{11}$$

Now, the force due to the line source is

$$\mathbf{F} = \rho\int_{S_0}\mathbf{\Gamma}\,dS + \rho\int_{S_1}\mathbf{\Gamma}\,dS + \rho\int_{S_2}\mathbf{\Gamma}\,dS, \tag{12}$$

so that, by using the expressions for $\mathbf{\Gamma}$ on S_0 and S_1 as given by (9) and (11) respectively, and from the fact that on these surfaces

$$dS = -2\pi\epsilon^2 \sin\theta \, d\theta,$$

$$\mathbf{F} = 2\pi\rho\mathbf{i}\gamma_0^2 \int_{\pi}^{\frac{1}{2}\pi} \frac{\sin\theta \, d\theta}{1-\cos\theta} - 2\pi\rho\mathbf{i}\gamma_a^2 \int_{\frac{1}{2}\pi}^{0} \frac{\sin\theta \, d\theta}{1+\cos\theta} + \rho \int_{S_2} \mathbf{\Gamma} \, dS + O(\epsilon \log\epsilon)$$

$$= 2\pi\rho\mathbf{i}(\gamma_a^2 - \gamma_0^2)\log 2 + \rho \int_{S_2} \mathbf{\Gamma} \, dS + O(\epsilon \log\epsilon). \tag{13}$$

6. The expression for $\mathbf{\Gamma}$ on the cylinder S_2

The position vector of a point on S_2 can be written as $\mathbf{r} = x\mathbf{i} + \epsilon\boldsymbol{\omega}$ where $0 \leqslant x \leqslant a$ and $\boldsymbol{\omega}$, as before, is a unit vector perpendicular to $\mathbf{i}$ and which on S_2 represents the unit normal $\mathbf{n}$ to the cylinder. It is impossible to find a suitable representation for $\mathbf{\Gamma}$ valid over the whole cylindrical surface. In fact I found it best to split this range into three parts S_{20}, S_{21}, S_{22} governed respectively by $0 \leqslant x \leqslant \delta$, $\delta \leqslant x \leqslant a-\delta$, and $a-\delta \leqslant x \leqslant a$, where δ is infinitesimally small and chosen to be $\epsilon^{\frac{1}{2}}$ and hence greater than ϵ, the radius of the section. The reason for this choice of δ (which will become more evident in the ensuing analysis) is that it certainly makes ϵ/x less than unity over the cylinder S_{21} so that r can be expressed as a binomial expansion in ϵ.

Taking first the range $0 \leqslant x \leqslant \delta = \epsilon^{\frac{1}{2}}$ it can be shown that

$$\log\left(\frac{r+r_1-a}{r+r_1+a}\right) = \log\left(\frac{r-x}{2a}\right) + O(\epsilon^{\frac{1}{2}}), \tag{14.1}$$

$$(\hat{\mathbf{r}}+\hat{\mathbf{r}}_1)\left(\frac{1}{r+r_1-a} - \frac{1}{r+r_1+a}\right) = \frac{\epsilon\boldsymbol{\omega}}{r(r-x)} - \frac{\mathbf{i}}{r} + O(1). \tag{14.2}$$

Hence, from (7), the expression for $\mathbf{q}$ on S_{20} is

$$\left(\frac{\epsilon\boldsymbol{\omega}}{r(r-x)} - \frac{\mathbf{i}}{r}\right)\phi_0 - \mathbf{q}_0 \log\left(\frac{r-k}{2a}\right) + O(1),$$

where ϕ_0 and $\mathbf{q}_0$ are to be evaluated at $\mathbf{r} = x\mathbf{i} + \epsilon\boldsymbol{\omega}$ $(0 \leqslant k \leqslant \delta)$.

Again, the expression for $\mathbf{\Gamma}$ will be

$$\mathbf{\Gamma} = \mathbf{q}(\boldsymbol{\omega}.\mathbf{q}) - \tfrac{1}{2}\boldsymbol{\omega}q^2$$

$$= \left\{\tfrac{1}{2}\boldsymbol{\omega}\left(\frac{\epsilon^2}{r^2(r-x)^2} - \frac{1}{r^2}\right) - \frac{\epsilon\mathbf{i}}{r^2(r-x)}\right\}\phi_0^2(x,r) + O(\epsilon^{-1}\log\epsilon). \tag{15}$$

Since for the elemental strips of the cylinder bounded by the planes $x =$ constant and $x+dx =$ constant, $\int \boldsymbol{\omega} \, dS = 0$, it follows that the

contribution of $\boldsymbol{\omega}$ to the integration of $\boldsymbol{\Gamma}$ over S_{20} is zero. Also, since, on S_{20}, $S = O(\epsilon^{\frac{1}{2}})$, $dS = 2\pi\epsilon\, dx$,

$$\int_{S_{20}} \boldsymbol{\Gamma}\, dS = -2\pi\epsilon^2 \mathbf{i} \int_0^\delta \frac{\phi_0^2(x,r)}{r^2(r-k)}\, dx + O(\epsilon^{\frac{1}{2}} \log \epsilon). \tag{16}$$

Since, as was stated in § 2, $\phi_0(x,r)$ is a function of r through r^2, then

$$\phi_0(x,r) = \mu(x) + O(\epsilon^2), \tag{17}$$

where $\mu(x) \equiv \phi(x,x)$. Furthermore, by expanding about $x = 0$, since $0 \leqslant x \leqslant \delta$,

$$\phi_0(x,r) = \mu(0) + x\mu'(0) + O(\epsilon). \tag{18}$$

It is not necessary to expand further since, in the given range of x, $r^{-2}(r-x)^{-1}$ has the maximum value of $3\sqrt{3}/4\epsilon^3$. Furthermore

$$\left.\begin{aligned} \epsilon^2 \int_0^\delta \frac{dx}{r^2(r-x)} &= \log\left\{\frac{\delta^2+\epsilon^2+\delta\sqrt{(\delta^2+\epsilon^2)}}{\epsilon^2}\right\} = -\log \tfrac{1}{2}\epsilon + O(\epsilon) \\ \text{and} \quad \epsilon^2 \int_0^\delta \frac{x\, dx}{r^2(r-x)} &= \sqrt{(\delta^2+\epsilon^2)} - \epsilon + \delta - \epsilon \tan^{-1}(\delta/\epsilon) = O(\epsilon^{\frac{1}{2}}) \end{aligned}\right\}. \tag{19}$$

Combining the results from (16) to (19) and using the notation in (9), viz. $\mu(0) = \phi(0,0) = \gamma_0$, we see that

$$\int_{S_{20}} \boldsymbol{\Gamma}\, dS = 2\pi\gamma_0^2 \mathbf{i} \log \tfrac{1}{2}\epsilon + O(\epsilon^{\frac{1}{2}} \log \epsilon). \tag{20}$$

From symmetrical considerations it follows that the integral of $\boldsymbol{\Gamma}$ over the third range $a-\delta \leqslant x \leqslant a$ is

$$\int_{S_{22}} \boldsymbol{\Gamma}\, dS = -2\pi\gamma_a^2 \mathbf{i} \log \tfrac{1}{2}\epsilon + O(\epsilon^{\frac{1}{2}} \log \epsilon). \tag{21}$$

It remains now to evaluate the integral of $\boldsymbol{\Gamma}$ over the central portion of the cylinder, viz. S_{21} described by $\delta \leqslant x \leqslant a-\delta$. Since I have chosen $\delta = \epsilon^{\frac{1}{2}} > \epsilon$, then r can be expressed as

$$r = \sqrt{(\epsilon^2+x^2)} = x + \tfrac{1}{2}\epsilon^2 x^{-1} - \tfrac{1}{8}\epsilon^4 x^{-3} + O(\epsilon^6),$$

with a similar expression for r_1. Hence it appears that

$$\log\left(\frac{r+r_1-a}{r+r_1+a}\right) = \log\left\{\frac{\epsilon^2}{4x(a-x)}\right\} + O(\epsilon), \tag{22.1}$$

$$(\hat{\mathbf{r}}+\hat{\mathbf{r}}_1)\left(\frac{1}{r+r_1-a} - \frac{1}{r+r_1+a}\right) = \frac{2\boldsymbol{\omega}}{\epsilon} + \frac{2x-a}{x(a-x)}\mathbf{i} + P(x)\boldsymbol{\omega} + R(x)\mathbf{i}, \tag{22.2}$$

where $$P(x) = -\tfrac{3}{2}\epsilon\{x^3+(a-x)^3\}/\{ax^2(a-x)^2\}$$

and $R(x)$ is a function of x which at the most is $O(\epsilon^{\frac{1}{2}})$.

By using (7), as before, it follows that, when $\delta \leqslant x \leqslant a-\delta$, the expression on S_{21} is

$$\mathbf{q} = \mathbf{Q} - \{2\log\epsilon - \log[4x(a-x)]\}\mathbf{q}_0 + \\ + \left\{\frac{2\boldsymbol{\omega}}{\epsilon} + \frac{2x-a}{x(a-x)}\mathbf{i} + P(x)\boldsymbol{\omega} + R(x)\mathbf{i}\right\}\phi_0 + \mathbf{q}_1 + O(\epsilon), \quad (23)$$

where $\mathbf{Q}$, $\mathbf{q}_0$, ϕ_0, $\mathbf{q}_1$, ϕ_1 are all to be evaluated at $\mathbf{r} = x\mathbf{i} + \epsilon\boldsymbol{\omega}$. As before, on this cylinder,

$$\left.\begin{aligned} \phi_0(x,r) &= \mu(x) + O(\epsilon^2), \quad \text{where } \mu(x) \equiv \phi_0(x,x) \\ \mathbf{Q}(x\mathbf{i} + \epsilon\boldsymbol{\omega}) &= \mathbf{Q}(x\mathbf{i}) + O(\epsilon) \end{aligned}\right\}. \quad (24)$$

Furthermore, since

$$\mathbf{q}_0 = -\operatorname{grad}\phi_0(x,r) = -\frac{\partial\phi_0}{\partial x}\mathbf{i} - \frac{\partial\phi_0}{\partial r}\hat{\mathbf{r}},$$

its value on the cylinder is simply

$$\mathbf{q}_0 = -\left(\frac{\partial\phi_0}{\partial x}\right)_{r=x}\mathbf{i} - \left(\frac{\partial\phi_0}{\partial r}\right)_{r=x}\mathbf{i} + O(\epsilon) = -\mu'(x)\mathbf{i} + O(\epsilon), \quad (25)$$

where the prime denotes differentiation in x. Similarly, if $\phi_1(x,x)$ is denoted by $\lambda(x)$, then on S_{21}

$$\mathbf{q}_1 = -\lambda'(x)\mathbf{i} + O(\epsilon). \quad (26)$$

Consequently, on S_{21}, the fluid velocity $\mathbf{q}$ takes the form

$$\mathbf{q} = \mathbf{t}_0\log\epsilon + \mathbf{t}_1\epsilon^{-1} + \mathbf{t}_2 + O(\epsilon^{\frac{1}{2}}), \quad (27)$$

where, on using the previous four relations,

$$\mathbf{t}_0 = 2\mu'(x)\mathbf{i}, \qquad \mathbf{t}_1 = 2\mu(x)\boldsymbol{\omega}, \qquad \mathbf{t}_2 = \mathbf{Q}(x\mathbf{i}) + L(x)\mathbf{i} + P(x)\boldsymbol{\omega}, \quad (28)$$

$L(x)$ being the function defined by

$$L(x) = -\mu'(x)\log\{4x(a-x)\} + \mu(x)(2x-a)/[x(a-x)] - \lambda'(x).$$

Now, on S_{21},

$$\boldsymbol{\Gamma} = \mathbf{q}(\boldsymbol{\omega}.\mathbf{q}) - \tfrac{1}{2}\boldsymbol{\omega}q^2 \\ = \boldsymbol{\Gamma}_1(\log\epsilon)^2 + \boldsymbol{\Gamma}_2\epsilon^{-1}\log\epsilon + \boldsymbol{\Gamma}_3\epsilon^{-2} + \boldsymbol{\Gamma}_4\epsilon^{-1} + O(\epsilon^{-\frac{1}{2}}), \quad (29.1)$$

where

$$\boldsymbol{\Gamma}_1 = (\boldsymbol{\omega}.\mathbf{t}_0)\mathbf{t}_0 - \tfrac{1}{2}\boldsymbol{\omega}t_0^2 = -2\mu'^2(x)\boldsymbol{\omega}, \quad (29.2)$$

$$\boldsymbol{\Gamma}_2 = (\boldsymbol{\omega}.\mathbf{t}_1)\mathbf{t}_0 + (\boldsymbol{\omega}.\mathbf{t}_0)\mathbf{t}_1 - (\mathbf{t}_0.\mathbf{t}_1)\boldsymbol{\omega} = 4\mu(x)\mu'(x)\mathbf{i}, \quad (29.3)$$

$$\boldsymbol{\Gamma}_3 = (\boldsymbol{\omega}.\mathbf{t}_1)\mathbf{t}_1 - \tfrac{1}{2}\boldsymbol{\omega}t_1^2 = 2\mu^2(x)\boldsymbol{\omega}, \quad (29.4)$$

$$\boldsymbol{\Gamma}_4 = (\boldsymbol{\omega}.\mathbf{t}_2)\mathbf{t}_1 + (\boldsymbol{\omega}.\mathbf{t}_1)\mathbf{t}_2 - (\mathbf{t}_1.\mathbf{t}_2)\boldsymbol{\omega} = 2\mu(x)\{\mathbf{Q}(x\mathbf{i}) + L(x)\mathbf{i} + P(x)\boldsymbol{\omega}\}. \quad (29.5)$$

Before proceeding further towards the final result I shall now identify the functions $\mu(x)$ and $\lambda(x)$ in terms of the density $f(x)$ of the source distribution.

7. The identification of $\mu(x)$ and $\lambda(x)$

It will be remembered that the functions $\mu(x)$ and $\lambda(x)$ were obtained from the functions $\phi_0(x,r)$ and $\phi_1(x,r)$ by putting $r = x$. These latter functions in turn are defined by the relation

$$\int_0^a \frac{f(\sigma)\,d\sigma}{|\mathbf{r}-\sigma\mathbf{i}|} = \phi_0(x,r)\log\left(\frac{r+r_1+a}{r+r_1-a}\right)+\phi_1(x,r) \tag{30}$$

on the cylinder $\mathbf{r} = x\mathbf{i}+\epsilon\boldsymbol{\omega}$, so that the right-hand side is

$$\mu(x)\{-2\log\epsilon+\log[4x(a-x)]\}+\lambda(x)+O(\epsilon^2). \tag{31}$$

By the first mean-value theorem

$$\left.\begin{aligned} f(\sigma) &= f(x)+(\sigma-x)g(x,\sigma) \\ \text{where}\qquad g(x,\sigma) &= f'\{\theta\lambda+(1-\theta)x\} \quad (0<\theta<1) \end{aligned}\right\}, \tag{32}$$

so that the left-hand side of (30) is

$$f(x)\int_0^a \frac{d\sigma}{|\mathbf{r}-\sigma\mathbf{i}|}+\int_0^a \frac{(\sigma-x)g(x,\sigma)}{|\mathbf{r}-\sigma\mathbf{i}|}\,d\sigma \tag{33}$$

$$= f(x)\{-2\log\epsilon+\log[4x(a-x)]\}+K+O(\epsilon^2), \tag{34}$$

where K denotes the second integral in (33) and is free from a logarithmic singularity at $\epsilon = 0$. Hence, on equating coefficients of $\log\epsilon$ in (31) and (34),

$$\mu(x) = f(x). \tag{35}$$

It follows also that $\lambda(x) = K+O(\epsilon^2)$ and, since $\lambda(u)$ is independent of ϵ, this relation can be replaced by

$$\lambda(x) = \lim_{\epsilon\to 0} K = \lim_{\epsilon\to 0}\int_0^a \frac{(\sigma-x)g(x,\sigma)\,d\sigma}{|(\sigma-x)\mathbf{i}+\epsilon\boldsymbol{\omega}|}. \tag{36}$$

If the range of integration $(0,a)$ is split into three parts by the closed intervals $(0,x-\eta)$, $(x-\eta,x+\eta)$, $(x+\eta,a)$, where $0<\eta<\epsilon$, then in the first range the integrand is $-1+O(\epsilon^2)$, whilst in the third range it is $1+O(\epsilon^2)$. Furthermore

$$L = \left|\int_{x-\eta}^{x+\eta}\frac{(\sigma-x)g(x,\sigma)\,d\sigma}{|(\sigma-x)\mathbf{i}+\epsilon\boldsymbol{\omega}|}\right| < \frac{\eta}{\epsilon}\int_{x-\eta}^{x+\eta}|g(x,\sigma)|\,d\sigma.$$

But $|g(x,\sigma)|$ is bounded in $(x-\eta,x+\eta)$ since, by (32), $g(x,\sigma)$ has the same radius of convergence as $f(x)$ which is greater than a. Hence, if we keep ϵ fixed, L will tend to zero with η, so that

$$K = \int_0^x -g(x,\sigma)\,d\sigma+\int_x^a g(x,\sigma)\,d\sigma+O(\epsilon^2).$$

Hence, by using (32) and (36),

$$\lambda(x) = \left\{\int_x^a + \int_x^0\right\} \frac{f(x)-f(\sigma)}{x-\sigma}\, d\sigma. \tag{37}$$

8. The final expression for F

I shall now integrate the $\mathbf{\Gamma}$ given in equations (29) over the portion of the cylinder S_{21}. It is obvious that $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_3$ contribute nothing to the integral of $\mathbf{\Gamma}$ and neither do the terms in $\boldsymbol{\omega}$ in $\mathbf{\Gamma}_4$. Since, on this surface, $dS = 2\pi\epsilon\, dx$,

$$\int_{S_{21}} \mathbf{\Gamma}_2\, dS = 8\pi\epsilon\mathbf{i} \int_\delta^{a-\delta} \mu(x)\mu'(x)\, dx = 4\pi\epsilon\mathbf{i}\{\mu^2(a-\delta)-\mu^2(\delta)\}$$
$$= 4\pi\epsilon\mathbf{i}(\gamma_a^2-\gamma_0^2)+O(\epsilon^2). \tag{38}$$

Also
$$\int_{S_{21}} \mathbf{\Gamma}_4\, dS = 4\pi\epsilon \int_\delta^{a-\delta} \mu(x)\{\mathbf{Q}(x\mathbf{i})-\lambda'(x)\mathbf{i}\}\, dx + 4\pi\epsilon\mathbf{i}I, \tag{39}$$

where

$$I = \int_\delta^{a-\delta} \left\{\frac{2x-a}{x(a-x)}\mu^2(x) - \log[4x(a-x)]\mu(x)\mu'(x)\right\} dx$$
$$= J - \tfrac{1}{2}\{\mu^2(a-\delta)-\mu^2(\delta)\}\log 4\delta(a-\delta)$$
$$= J - \tfrac{1}{2}(\gamma_a^2-\gamma_0^2)\log 4\delta(a-\delta)+O(\delta\log\delta) \tag{40}$$

and

$$J = \int_\delta^{a-\delta} \tfrac{1}{2}\mu^2(x)\left\{\frac{1}{a-x}-\frac{1}{x}\right\} dx$$
$$= \tfrac{1}{2}(\gamma_a^2-\gamma_0^2)\log[(a-\delta)/\delta]+H(\delta), \tag{41}$$

where
$$H(\delta) = \frac{1}{2}\int_\delta^{a-\delta} \left\{\frac{\mu^2(x)-\mu^2(a)}{a-x} - \frac{\mu^2(x)-\mu^2(0)}{x}\right\} dx, \tag{42}$$

so that $H(\delta)$ is $O(1)$ when $\delta \to 0$.

Hence, from (29) and (38)–(42),

$$\int_{S_{21}} \mathbf{\Gamma}\, dS = 4\pi \int_\delta^{a-\delta} \mu(x)\{\mathbf{Q}(x\mathbf{i})-\lambda'(x)\mathbf{i}\}\, dx +$$
$$+ 4\pi\mathbf{i}H(\delta)+2\pi\mathbf{i}(\gamma_a^2-\gamma_0^2)\log\tfrac{1}{4}\epsilon+O(\delta\log\delta). \tag{43}$$

Combining this result with (13), (20), (21) we can see that the logarithmic singularity at $\epsilon = 0$ disappears leaving the force $\mathbf{F}$ as

$$\mathbf{F} = 4\pi\rho \int_\delta^{a-\delta} \mu(x)\{\mathbf{Q}(x\mathbf{i})-\lambda'(x)\mathbf{i}\}\, dx + 4\pi\rho H(\delta)\mathbf{i}+O(\delta\log\delta).$$

If now δ is made to tend to zero, i.e. the tight surface shrinks into the position of the line source, the expression for $\mathbf{F}$, in terms of $f(x)$, etc., by using (35) and (37) can be written as

$$\mathbf{F} = 4\pi\rho \int_0^a f(x)\mathbf{Q}(x\mathbf{i})\,dx + 4\pi\rho\mathbf{i}R(x),$$

where
$$R(x) = H(0) - \int_0^a f(x)\lambda'(x)\,dx$$

and
$$\lambda(x) = \left\{\int_x^a + \int_x^0\right\}\frac{f(x)-f(\sigma)}{x-\sigma}\,d\sigma.$$

If we integrate by parts the integral in the expression for $R(x)$ and use (42), $\mathbf{F}$ can finally be written as

$$\left.\begin{aligned}\mathbf{F} &= 4\pi\rho\int_0^a \{f(x)\mathbf{Q}(x\mathbf{i}) + T(x)\mathbf{i}\}\,dx \\ \text{where}\quad T(x) &= \frac{1}{2}\left\{\frac{[f(x)-f(a)]^2}{a-x} - \frac{[f(x)-f(0)]^2}{x}\right\} + f'(x)\left\{\int_x^a + \int_x^0\right\}\frac{f(x)-f(\sigma)}{x-\sigma}\,d\sigma\end{aligned}\right\}. \quad (44)$$

9. An expression for the moment

The moment of the hydrodynamic forces on the obstacle taken about the origin is simply

$$\mathbf{G}_0 = \rho \int_{(S_0+S_1+S_2)} \mathbf{r}\wedge\mathbf{\Gamma}\,dS, \quad (45)$$

where the expressions for $\mathbf{\Gamma}$ on S_0 and S_1 are given by (9) and (11) respectively, and on the three sections by (15), by its symmetrical equivalent, and by the relation (29).

Since, on S_0, $\mathbf{r} = O(\epsilon)$, $\mathbf{\Gamma} = O(\epsilon^{-2})$, and $S_0 = O(\epsilon^2)$, the contribution to $\mathbf{G}_0$ from S_0 tends to zero with ϵ. On S_1, $\mathbf{r} = a\mathbf{i}+\epsilon\mathbf{n}$ and $\mathbf{\Gamma}$ is given by (11). Hence, on S_1, $\mathbf{r}\wedge\mathbf{\Gamma} = O(\epsilon^{-1}\log\epsilon)$ so that, since $S_1 = O(\epsilon^2)$, there is no contribution to $\mathbf{G}_0$ from S_1 when ϵ tends to zero. Again, from (15), it appears that, on S_{20},

$$\mathbf{r}\wedge\mathbf{\Gamma} = \left\{\frac{x}{2r^2}\left(\frac{\epsilon^2}{(r-x)^2}-1\right) + \frac{\epsilon^2}{r^2(r-x)}\right\}(\mathbf{i}\wedge\boldsymbol{\omega})\phi_0^2(x,r) + O(\epsilon^{-\frac{1}{2}}\log\epsilon),$$

i.e.
$$\int_{S_{20}} \mathbf{r}\wedge\mathbf{\Gamma}\,dS = O(\epsilon^{\frac{3}{2}}\log\epsilon) \to 0 \quad \text{as } \epsilon \to 0.$$

Similarly it can be shown that

$$\int_{S_{22}} \mathbf{r} \wedge \mathbf{\Gamma}\, dS \to 0$$

with ϵ like $\epsilon \log \epsilon$. Hence the only contribution, if any, to $\mathbf{G}_0$ comes entirely from S_{21}. On this surface $\mathbf{r} = x\mathbf{i} + \epsilon\boldsymbol{\omega}$ ($\delta \leqslant x \leqslant a - \delta$) and using $\mathbf{\Gamma}$ as given by (29) we have

$$\mathbf{r} \wedge \mathbf{\Gamma} = \{2x\mu'^2(x)(\log\epsilon)^2 + 4\mu(x)\mu'(x)\log\epsilon - 2x\mu^2(x)\epsilon^{-2} - 2x\mu(x)P(x)\epsilon^{-1}\}(\boldsymbol{\omega} \wedge \mathbf{i}) + 2x\mu(x)\mathbf{i} \wedge \mathbf{Q}(x\mathbf{i})\epsilon^{-1} + O(\log\epsilon).$$

The term in $\boldsymbol{\omega} \wedge \mathbf{i}$ vanishes on integrating over the surface and, since on this surface $dS = 2\pi\epsilon\, dx$,

$$\mathbf{G}_0 = 4\pi\rho \int_{\delta}^{a-\delta} x\mu(x)\mathbf{i} \wedge \mathbf{Q}(x\mathbf{i})\, dx + O(\epsilon \log \epsilon).$$

Now allowing $\epsilon \to 0$ and remembering that $\mu(x) \equiv f(x)$ we get

$$\mathbf{G}_0 = 4\pi\rho \int_0^a x f(x)\mathbf{i} \wedge \mathbf{Q}(x\mathbf{i})\, dx. \tag{46}$$

10. The combination of the line source with an end-point source

In order to work out the examples I have chosen I need to find an expression for the force due to a combination of the previous line source with a point source of strength m at the end point $\mathbf{r} = a\mathbf{i}$. The technique used will serve as an example of the procedure to be used in similar problems.

The combination of line source and end source can be obtained as the limiting position of the line source between $\mathbf{r} = 0$ and $a\mathbf{i}$ and the point source of strength m at $\mathbf{r} = (a+\epsilon)\mathbf{i}$ ($\epsilon > 0$) when $\epsilon \to 0$. Since $\epsilon > 0$, the point source does not intersect the line source, so that the fluid velocity $\mathbf{q}$ is made up of three contributions, viz. $\mathbf{Q}_L$ due to the line source, $\mathbf{Q}_S$ due to the point source, and $\mathbf{Q}$ due to all other effects. Then the contribution to the force on an obstacle due to the line source and point source is $\mathbf{F}$ defined by

$$\mathbf{F}/(4\pi\rho) = \int_0^a \{f(x)\mathbf{Q}(x\mathbf{i}) + T(x)\mathbf{i}\}\, dx + \int_0^a f(x)\mathbf{Q}_S(x\mathbf{i})\, dx + m\mathbf{Q}_L\{(a+\epsilon)\mathbf{i}\} + m\mathbf{Q}\{(a+\epsilon)\mathbf{i}\}, \tag{47}$$

where $T(x)$ is defined by (44) and

$$\mathbf{Q}_S(x\mathbf{i}) = m(x-a-\epsilon)\mathbf{i}/|x-a-\epsilon|^3 = -m\mathbf{i}/(a+\epsilon-x)^2 \qquad (48)$$

when $0 \leqslant x \leqslant a$. Expanding $f(x)$ in the neighbourhood of $x = a$ by the second mean-value theorem gives

$$f(x) = f(a)+(x-a)f'(a)+(x-a)^2\nu(x,a). \qquad (49)$$

Hence
$$\int_0^a f(x)\mathbf{Q}_S(x\mathbf{i})\,dx = -m\mathbf{i}\Xi,$$

where, by (49),

$$\left.\begin{aligned}\Xi = \int_0^a \frac{f(x)\,dx}{(a+\epsilon-x)^2} = f(a)\left(\frac{1}{\epsilon}-\frac{1}{a}\right)+f'(a)\left(\log\frac{\epsilon}{a}+1\right)+ \\ +\int_0^a \frac{(x-a)^2\nu(x,a)}{(a+\epsilon-x)^2}\,dx+O(\epsilon)\end{aligned}\right\}. \qquad (50)$$

Again, to find $\mathbf{Q}_L\{(a+\epsilon)\mathbf{i}\}$ put $\mathbf{r} = (a+\epsilon)\mathbf{i}$, $\mathbf{r}_1 = \epsilon\mathbf{i}$, $\mathbf{Q} = 0$ in (7), so that $\mathbf{Q}_L\{(a+\epsilon)\mathbf{i}\} = \mathbf{i}Q_L$, where

$$Q_L = f(a)\left(\frac{1}{\epsilon}-\frac{1}{a}\right)+f'(a)\left(\log\frac{\epsilon}{a}+1\right)-\lambda'(a)+O(\epsilon\log\epsilon), \qquad (51)$$

where $\lambda(x)$ is defined by (37). Consequently, from (47), (50), (51)

$$\mathbf{F}/(4\pi\rho) = \int_0^a \{f(x)\mathbf{Q}(x\mathbf{i})+T(x)\mathbf{i}\}\,dx - \\ -m\mathbf{i}\int_0^a \frac{(x-a)^2\nu(x,a)\,dx}{(a+\epsilon-x)^2}-m\mathbf{i}\lambda'(a)+m\mathbf{Q}(a\mathbf{i})+O(\epsilon\log\epsilon).$$

If now $\epsilon \to 0$, so that the point source attains its limiting position at $\mathbf{r} = a\mathbf{i}$,

$$\mathbf{F}/(4\pi\rho) = \int_0^a \{f(x)\mathbf{Q}(x\mathbf{i})+T(x)\mathbf{i}\}\,dx - m\mathbf{i}\left(\lambda'(a)+\int_0^a \nu(x,a)\,dx\right)+m\mathbf{Q}(a\mathbf{i}). \qquad (52)$$

However, from (37),

$$-\lambda'(a) = \int_0^a \left[\frac{\partial}{\partial x}\left\{\frac{f(x)-f(\sigma)}{x-\sigma}\right\}\right]_{x=a} d\sigma + 2f'(a)$$

$$= \int_0^a \nu(\sigma,a)\,d\sigma+2f'(a),$$

so that the final expression for $\mathbf{F}$ is

$$\mathbf{F}/(4\pi\rho) = \int_0^a \{f(x)\mathbf{Q}(x\mathbf{i}) + T(x)\mathbf{i}\}\, dx + 2m\mathbf{i}f'(a) + m\mathbf{Q}(a\mathbf{i}). \tag{53}$$

11. The mutual interference of n airship forms placed in line in a uniform stream

Consider n line sinks arranged in line and in order along the $\mathbf{i}$-axis, the kth of which ($k = 1, 2, \ldots, n$) lies between the points $a_k\mathbf{i}$ and $b_k\mathbf{i}$ ($b_k > a_k$) and is a uniform distribution having strength $m_k/(b_k - a_k)$ per unit length and possessing an isolated source of strength m_k at each of the end points $b_k\mathbf{i}$ ($k = 1, 2, \ldots, n$) respectively.

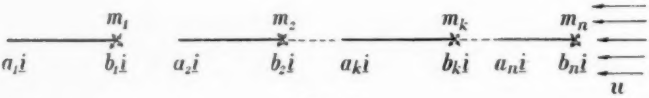

FIG. 3.

If a uniform stream $-U\mathbf{i}$ is introduced, the Stokes stream function of the flow is given by

$$\psi = \sum_{s=1}^{n} \left\{ \frac{m_s(x - b_s)}{r_s} + \frac{m_s(r_s - R_s)}{b_s - a_s} \right\} + \tfrac{1}{2}U\omega^2, \tag{54}$$

where $r_s = |\mathbf{r} - b_s\mathbf{i}|$, $R_s = |\mathbf{r} - a_s\mathbf{i}|$, ω being the polar cylindrical co-ordinate.

It can be shown that the dividing streamline $\psi = 0$ consists of the $\mathbf{i}$-axis and n bodies of revolution each enclosing one of the line singularities when the m's are small enough compared with the length $b_n - a_1$. A sufficient set of conditions is

$$\left.\begin{aligned} &\Delta_k < 1 \quad (k = 2, \ldots, n-1), \\ &\text{where} \\ &\Delta_k = \sum_{s=1}^{n} 8m_s(b_s - a_s)/[U(a_k + b_{k-1} - 2b_s)^2 |a_k + b_{k-1} - 2a_s|] \end{aligned}\right\}. \tag{55}$$

In this case the velocity at large r is simply $\mathbf{q} = \mathbf{U} + O(r^{-3})$, so that the effect of the envelope (discussed in the introduction) on the force is zero. Consequently by using the result (53) it can be seen that the force $\mathbf{F}_k$ which the body of revolution surrounding (a_k, b_k) experiences is simply

$$\mathbf{F}_k/(4\pi\rho) = -\frac{m_k}{(b_k - a_k)} \int_{a_k}^{b_k} \mathbf{Q}(x\mathbf{i})\, dx + m_k\, \mathbf{Q}(b_k\mathbf{i}). \tag{56}$$

Since $T(x) = 0$, $f'(a) = 0$ when $f(x) = \text{constant} = -m_k/(b_k - a_n)$ and $\mathbf{Q}(x\mathbf{i})$ is the fluid velocity due to all contributions excepting the line sink and point source between (a_k, b_k). Furthermore, it can be shown that

$$\mathbf{Q}(x\mathbf{i}) = \mathbf{i} \sum_{s=1}^{n}{}^{*} \frac{(k-s)}{|k-s|} m_s \left\{ \frac{1}{(x-b_s)^2} - \frac{1}{(x-b_s)(x-a_s)} \right\}, \tag{57}$$

where the asterisk in the summation denotes the exclusion of $s = k$. Hence, by (56), the final expression for $\mathbf{F}_k$ is

$$\mathbf{F}_k = 4\pi\rho m_k \sum_{s=1}^{n}{}^{*} m_s \frac{(k-s)}{|k-s|} \left[\frac{1}{C_{kk} C_{ss}} \log\left(\frac{B_{ks} A_{ks}}{C_{ks} C_{sk}} \right) - \frac{1}{B_{ks}} \left(\frac{1}{C_{ks}} + \frac{1}{C_{sk}} - \frac{1}{B_{ks}} \right) \right], \tag{58}$$

where

$$A_{ks} = A_{sk} = |a_k - a_s|, \qquad B_{ks} = B_{sk} = |b_k - b_s|,$$
$$C_{ks} = |b_k - a_s|, \qquad C_{sk} = |b_s - a_k|.$$

It is well to remember that $\sum_{k=1}^{n} \mathbf{F}_k = 0$.

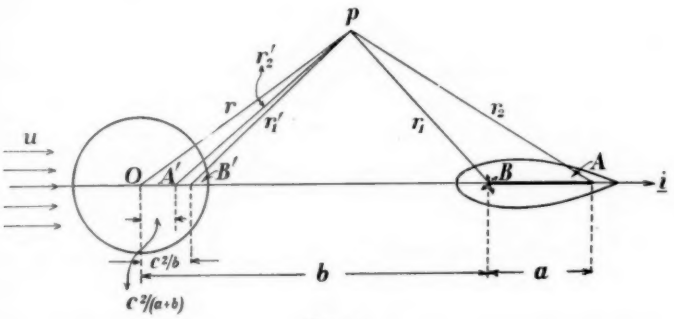

FIG. 4.

12. The force on an airship form in a uniform stream due to a sphere

Another interesting example can be constructed using Weiss's theorem (**2**).

The velocity potential ϕ due to a uniform stream $U\mathbf{i}$, a source of strength m at $b\mathbf{i}$, a uniform line sink between $b\mathbf{i}$ and $(a+b)\mathbf{i}$ of total strength m in the presence of a sphere $|\mathbf{r}| = c$, where $c < b$, can be written down, by using Weiss's theorem and the identity

$$r|c^2\mathbf{r}r^{-2}\sigma - \lambda\mathbf{i}| \equiv \lambda|\mathbf{r} - c^2\sigma\lambda^{-1}\mathbf{i}|,$$

as

$$\phi = -Ux+\frac{m}{r_1}-\frac{m}{a}\int_b^{a+b}\frac{d\lambda}{|\mathbf{r}-\lambda\mathbf{i}|}-\tfrac{1}{2}U\frac{c^3x}{r^3}+\frac{mc}{br_1}-\frac{mc}{b}\int_0^1\frac{d\sigma}{|\mathbf{r}-c^2b^{-1}\sigma\mathbf{i}|}-$$

$$-\frac{mc}{a}\int_b^{a+b}\frac{d\lambda}{\lambda|\mathbf{r}-c^2\lambda^{-1}\mathbf{i}|}+\frac{mc}{a}\int_0^1 d\sigma\int_b^{a+b}\frac{d\lambda}{\lambda|\mathbf{r}-c^2\sigma\lambda^{-1}\mathbf{i}|}. \quad (59)$$

The corresponding Stokes stream function ψ can be written immediately as

$$\psi = -\tfrac{1}{2}U\omega^2+\frac{m}{r_1}(x-b)-\frac{m}{a}\int_b^{a+b}\frac{(x-\lambda)\,d\lambda}{\lambda|\mathbf{r}-\lambda\mathbf{i}|}+\tfrac{1}{2}Uc^3\frac{\omega^2}{r^3}+\frac{mc}{br_1}(x-c^2b^{-1})-$$

$$-\frac{mc}{b}\int_0^1\frac{(x-c^2b^{-1}\sigma)\,d\sigma}{|\mathbf{r}-c^2b^{-1}\sigma\mathbf{i}|}-\frac{mc}{a}\int_b^{a+b}\frac{(x-c^2\lambda^{-1})\,d\lambda}{\lambda|\mathbf{r}-c^2\lambda^{-1}\mathbf{i}|}+$$

$$+\frac{mc}{a}\int_0^1 d\sigma\int_b^{a+b}\frac{(x-c^2\sigma\lambda^{-1})\,d\lambda}{\lambda|\mathbf{r}-c^2\sigma\lambda^{-1}\mathbf{i}|}.$$

If now $\overrightarrow{OP}=\mathbf{r}$ and A' and B' are the inverse points of A and B respectively in the sphere $|\mathbf{r}|=c$, then, denoting PA, PB, PA', PB' by r_2, r_1, r_2', r_1' respectively, we simplify the expression for ψ to

$$\psi = -\tfrac{1}{2}U\omega^2\left(1-\frac{c^3}{r^3}\right)+\frac{m}{r_1}(x-b)+$$

$$+\frac{mc}{b}\left(\frac{x-c^2b^{-1}}{r_1'}\right)-\frac{m}{a}(r_1-r_2)+\frac{m}{ac}(a+b)(r_1'-r_2'). \quad (60)$$

It can now be verified that $\psi=0$ when $r=c$ or $\omega=0$. Furthermore, when b is sufficiently large compared with c, we get a closed airship form as part of the dividing stream function $\psi=0$. The force on the sphere is given, by (53), as

$$\mathbf{F} = 4\pi m\rho\left(\mathbf{Q}(b\mathbf{i})-\frac{1}{a}\int_b^{a+b}\mathbf{Q}(x\mathbf{i})\,dx\right), \quad (61)$$

$Q(x\mathbf{i})$, as before, denoting the fluid velocity at $x\mathbf{i}$ due to all singularities excluding those in AB, i.e. calculated from the stream function

$$\psi = \tfrac{1}{2}Uc^3\frac{\omega^2}{r^3}+\frac{mc}{b}\left(\frac{x-c^2b^{-1}}{r_1'}\right)+\frac{m}{ac}(a+b)(r_1'-r_2').$$

To save space I quote the result of this calculation as

$$\mathbf{F} = -4\pi m\rho\mathbf{i}\left\{\frac{Uac^3(2a+3b)}{2b^3(a+b)^2} + \frac{mc\{b^3-(a+b)c^2\}}{(b^2-c^2)^2\{(a+b)b-c^2\}} - \right.$$
$$\left. - \frac{m(a+b)}{a^2c}\log\left[\left(\frac{(a+b)b-c^2}{(a+b)^2-c^2}\right)\left(\frac{(a+b)b-c^2}{b^2-c^2}\right)\right]\right\}. \quad (62)$$

Since there are no singularities outside the two bodies and since the effect of the envelope at infinity is zero in this case, this $\mathbf{F}$ does represent the force on the sphere and also $-\mathbf{F}$ is the force experienced by the airship form. When b is very large, the asymptotic form for $\mathbf{F}$ is simply

$$\mathbf{F} \sim -6\pi m\rho Uc^3ab^{-4}\mathbf{i}, \quad (63)$$

so that the two bodies repel one another.

Numerous other problems and their solutions can be constructed on similar lines to these.

In conclusion I wish to thank Professors V. C. A. Ferraro and A. T. Price for the kind interest they have taken in this paper and I am grateful to them for their helpful advice.

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ON THE INHOMOGENEOUS MINIMA OF CERTAIN CONVERGENT SEQUENCES OF BINARY QUADRATIC FORMS

By H. J. GODWIN (*Swansea*)

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1. LET $f(x, y)$ be an indefinite binary quadratic form and let $M(a, b)$ denote $\min f(x, y)$ for all values of x and y congruent modulo 1 to the given numbers a and b respectively. Let $M_1(f)$ be the least upper bound of $M(a, b)$ for all values of a and b. Then $M_1(f)$ is called the *inhomogeneous* minimum of the form f and is said to be *taken at* (a, b) if $M(a, b) = M_1(f)$. This quantity has been evaluated for a number of forms [see Barnes and Swinnerton-Dyer (**1**), where a bibliography is given]. If we exclude a certain set of points (a, b), then in many cases the least upper bound of $M(a, b)$ in the remaining set is less than $M_1(f)$ by a definite amount and is called the *second minimum* $M_2(f)$. For a few forms it has been shown [Davenport (**3**), Varnavides (**4**), Barnes and Swinnerton-Dyer (**2**)] that there exists an enumerably infinite set of discrete minima with a limit point below which the minima form a non-enumerable set.

The problem may be expressed geometrically as follows. Let

$$f(x, y) = \xi(x, y)\eta(x, y),$$

where ξ and η are linear functions of x and y. To the integral lattice in the (x, y)-plane corresponds a lattice in the (ξ, η)-plane; each point of this lattice is made the centre of a hyperbolic region $|\xi\eta| \leqslant K$ and $M_1(f)$ is the greatest lower bound of those values of K for which the regions cover the plane. If there is an infinity of lattice points with ξ or η coordinates arbitrarily near to any given one (as in the cases cited above), then the set of points left uncovered when K is less than $M_1(f)$ is nowhere dense, no matter how small K is. If, however, the coefficients of x and y in ξ and η are rational, then the values of ξ and η at the lattice points have only a finite number of residues modulo 1 and for values of K less than $M_1(f)$ the set of points left uncovered consists of a number of domains: consequently the set of minima is connected.

In this paper $M_1(f)$ is evaluated for each of a sequence of forms tending to a form for which the set of discrete minima is known; this is done for two cases, the limit forms being x^2+xy-y^2 [see Davenport (**3**)] and x^2-2y^2 [see Varnavides (**4**)]. If the sequence of forms is $f_1, f_2, \ldots$ converging to f and if $M_1(f_n)$ is taken at (a_n, b_n), then it is found that the

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limit points of the sequence (a_n, b_n) are points at which $M_1(f)$ is taken, while for each sequence for which (a_n, b_n) converges to a limit point the values of $M_1(f_n)$ converge to a value lying between $M_1(f)$ and $M_2(f)$.

A referee who read the first draft of this paper has pointed out that, if $M_1(f)$ is taken at (a, b) and (x, y) is sufficiently near to (a, b), then for sufficiently large n,

$$M(f; a, b) > M(f_n; x, y) - \epsilon$$

so that, if $\liminf M_1(f_n) > M_2(f)$, then we must have

$$\liminf M_1(f_n) \leqslant M_1(f),$$

and also the limit points of sequences (a_n, b_n) at which $M_1(f_n)$ is taken must be points at which $M_1(f)$ is taken. He has also told me that Dr. C. A. Rogers has proved that under certain conditions (which obtain in the cases discussed in this paper) $\limsup M_1(f_n) < M_1(f)$. For these comments and information I should like to express my gratitude.

2. In § 3 we denote by q_n the denominators of the convergents to the continued fraction $[1, 1, \ldots] = \frac{1}{2}(1+\sqrt{5}) = \omega$,

so that $q_0 = q_1 = 1$ and $q_n = q_{n-1} + q_{n-2}$.

This gives $q_n = \{\omega^{n+1} - (-\omega)^{-n-1}\}/\sqrt{5}$,

and it may be verified that

$$q_a = (-1)^a q_{-a-2}, \tag{1}$$

$$q_a q_b - q_{a-1} q_{b+1} = (-1)^a q_{b-a}, \tag{2}$$

$$q_a q_b + q_{a+1} q_{b+1} = q_{a+b+2}, \tag{3}$$

$$q_{a+3} + q_a = 2q_{a+2}, \tag{4}$$

$$q_{a+3} - q_a = 2q_{a+1}, \tag{5}$$

$$5q_a q_b = q_{a+b+2} + q_{a+b} + (-1)^a (q_{b-a} + q_{b-a-2}), \tag{6}$$

$$q_{a+4} + q_{a+2} + q_{a-2} + q_{a-4} = 10q_a, \tag{7}$$

$$q_{a+3} - q_{a-3} = 4q_a. \tag{8}$$

We have also

LEMMA 1. *The sequence q_{a+2c}/q_{b+2c} $(c = 1, 2, \ldots)$ is strictly increasing if $b > a$ and a is odd or if $b < a$ and b is even, and strictly decreasing if $b > a$ and a is even or if $b < a$ and b is odd $(b \geqslant -2)$.*

Proof. From (2),

$$\begin{aligned} q_{a+2c+2} q_{b+2c} - q_{a+2c} q_{b+2c+2} &= (-1)^a (q_{b-a-2} - q_{b-a}) \\ &= (-1)^{a-1} q_{b-a-1} \\ &= (-1)^b q_{a-b-1}, \quad \text{by (1).} \end{aligned}$$

Also q_{b+2c}, q_{b+2c+2} are positive.

In § 4 we denote by p_n, q_n the numerators and denominators of the convergents to the continued fraction

$$[1, 2, 2, 2, \ldots] = \sqrt{2}.$$

If $\rho = \sqrt{2}+1$, then

$$p_n = \{\rho^{n+1}+(-\rho)^{-n-1}\}/2, \qquad q_n = \{\rho^{n+1}-(-\rho)^{-n-1}\}/2\sqrt{2},$$

and it may be verified that (1), (2), and (3) still hold, while we have also

$$p_a = q_{a+1}-q_a = q_a+q_{a-1}, \tag{9}$$

$$p_a p_{2b-a}-p_b^2 = (-1)^{a-1}2q_{b-a-1}^2, \tag{10}$$

$$2q_a q_b-p_a p_b = (-1)^b p_{a-b-1}, \tag{11}$$

$$2q_a q_b+p_a p_b = p_{a+b+1}. \tag{12}$$

We need also a lemma on continued fractions.

LEMMA 2. *Let the convergents to the continued fraction for θ be*

$$p_0/q_0, p_1/q_1, \ldots$$

and let $\{r\theta\}$ denote the difference between $r\theta$ and the integer nearest to it, i.e. $\{r\theta\} = |r\theta-[r\theta+\frac{1}{2}]|$. Then, for $n \geqslant 2$, $\{r\theta\} < \{q_n\theta\}$ if and only if r equals q_{n+1}, $q_{n+1}+q_n$, $2q_{n+1}$, $2q_{n+1}+q_n, \ldots$, q_{n+2}, $q_{n+2}+q_{n+1}$, $q_{n+2}+q_{n+1}+q_n$, etc. If θ is rational and equals p_N/q_N, we require $n \leqslant N-2$.

Proof. Let $\theta = [a_0, a_1, \ldots]$, $\epsilon_n = [0, a_{n+1}, a_{n+2}, \ldots]$. Then

$$\theta = \frac{p_n+\epsilon_n p_{n-1}}{q_n+\epsilon_n q_{n-1}}, \qquad \{q_n\theta\} = |q_n\theta-p_n| = \frac{\epsilon_n}{q_n+\epsilon_n q_{n-1}}.$$

If $|r\theta-s| < \{q_n\theta\}$, then

$$|r(p_n+\epsilon_n p_{n-1})-s(q_n+\epsilon_n q_{n-1})| < \epsilon_n.$$

Hence rp_n-sq_n, $rp_{n-1}-sq_{n-1}$ are of opposite sign; if we put

$$|rp_n-sq_n| = \alpha, \qquad |rp_{n-1}-sq_{n-1}| = \beta,$$

then $r = \alpha q_{n-1}+\beta q_n$, and we require to find α, β such that $|\alpha-\beta\epsilon_n| < \epsilon_n$, i.e.

$$\left|\alpha\left(a_{n+1}+\frac{1}{a_{n+2}+\ldots}\right)-\beta\right| < 1.$$

This gives the following table of values for α, β, r.

α	β	r
1	a_{n+1}	q_{n+1}
1	$a_{n+1}+1$	$q_{n+1}+q_n$
2	$2a_{n+1}$	$2q_{n+1}$
2	$2a_{n+1}+1$	$2q_{n+1}+q_n$
. . . . .	. . . . .	. . . . .
a_{n+2}	$a_{n+2}a_{n+1}$	$a_{n+2}q_{n+1}$
a_{n+2}	$a_{n+2}a_{n+1}+1$	q_{n+2}
$a_{n+2}+1$	$a_{n+1}(a_{n+2}+1)+1$	$q_{n+2}+q_{n+1}$
$a_{n+2}+1$	$a_{n+1}(a_{n+2}+1)+2$	$q_{n+2}+q_{n+1}+q_n$, etc.,

which completes the proof.

I shall say that two regions

$$|(\xi-\xi_1)(\eta-\eta_1)| \leqslant K, \qquad |(\xi-\xi_2)(\eta-\eta_2)| \leqslant K$$

cover between their centres if they cover the rectangle whose sides are $\xi = \xi_1$, $\xi = \xi_2$, $\eta = \eta_1$, $\eta = \eta_2$. The condition for this is

$$K \geqslant \tfrac{1}{4}|\xi_1-\xi_2||\eta_1-\eta_2|.$$

If strict inequality holds here, I say that the regions *completely cover between their centres.*

For conciseness of expression the terms 'above', 'to the right of', etc., will be used in an obvious way about configurations in the (ξ, η)-plane.

3. In this section we denote by $H(x_0, y_0)$ the region

$$\left|(x-x_0)+\frac{q_{n+1}}{q_n}(y-y_0)\right|\left|(x-x_0)-\frac{q_{n-1}}{q_n}(y-y_0)\right| \leqslant K,$$

by $H_1(\nu)$ the region $H\{\frac{1}{2}+\frac{1}{2}(-1)^{\nu-1}q_{3\nu},\ \frac{1}{2}(-1)^{\nu}q_{3\nu-1}\}$,

by $H_2(\nu)$ the region $H\{\frac{1}{2}(-1)^{\nu-1}q_{3\nu-1},\ \frac{1}{2}+\frac{1}{2}(-1)^{\nu-1}q_{3\nu}\}$

(note that $q_{3\nu-1}$ is even and $q_{3\nu}$ odd).

by R_1 the rectangle

$$0 \leqslant \xi = x+(q_{n+1}/q_n)y \leqslant \tfrac{1}{2}, \quad 0 \leqslant \eta = x-(q_{n-1}/q_n)y \leqslant 1,$$

and by R_2 the rectangle $0 \leqslant \xi \leqslant q_{n+1}/q_n$, $-q_{n-1}/2q_n \leqslant \eta \leqslant 0$.

By (1), (2), (3) the centre of $H_1(\nu)$ is, in (ξ, η)-coordinates,

$$\left(\frac{q_n-q_{n-3\nu}}{2q_n}, \frac{q_n+(-1)^{\nu-1}q_{n+3\nu}}{2q_n}\right),$$

and the centre of $H_2(\nu)$ is

$$\left(\frac{q_{n+1}+(-1)^{\nu-1}q_{n+3\nu+1}}{2q_n},\ \frac{-q_{n-1}+q_{n-3\nu-1}}{2q_n}\right).$$

We begin by proving some lemmas on the covering of R_1 and R_2 by these regions.

LEMMA 3. *If $K \geqslant 21/100$, then adjacent members of the sequence $H_1(1),\ldots, H_1\{[\frac{1}{3}(n-1)]\}$ overlap without gaps in R_1.*

Proof. The region

$$|\xi-\xi_0||\eta-\tfrac{1}{2}-\eta_0| \leqslant K$$

covers the strip

$$\xi_0-\frac{K}{\frac{1}{2}+|\eta_0|} \leqslant \xi \leqslant \xi_0+\frac{K}{\frac{1}{2}+|\eta_0|} \quad (0 \leqslant \eta \leqslant 1).$$

Hence the required condition is

$$\frac{2Kq_n}{q_n+q_{n+3\nu}}+\frac{2Kq_n}{q_n+q_{n+3\nu+3}} \geqslant \frac{q_{n-3\nu}-q_{n-3\nu-3}}{2q_n}$$

for $\nu = 1,\ldots, [\frac{1}{3}(n-4)]$.

By (4) and (5) this is equivalent to

$$K \geqslant \frac{q_{n-3\nu-2}(q_n+q_{n+3\nu})(q_n+q_{n+3\nu+3})}{4q_n^2(q_n+q_{n+3\nu+2})}.$$

This expression is less than

$$\begin{aligned}
&\frac{q_{n-3\nu-2}(q_n+q_{n+3\nu})q_{n+3\nu+3}}{4q_n^2 q_{n+3\nu+2}} \\
&\leqslant \frac{q_{11}+q_8}{q_{11}}\cdot\frac{q_{n-3\nu-2}q_{n+3\nu}q_{n+3\nu+3}}{4q_n^2 q_{n+3\nu+2}}, \quad \text{by Lemma 1, since } \nu \geqslant 1,\ n \geqslant 7, \\
&\qquad\qquad \text{and } q_7/q_{10} < q_8/q_{11}, \text{ by (2)}, \\
&\leqslant \frac{q_{11}+q_8}{q_{11}}\cdot\frac{q_{14}}{q_{13}}\cdot\frac{q_{n-3\nu-2}q_{n+3\nu}}{4q_n^2}, \quad \text{by Lemma 1}, \\
&= \frac{q_{11}+q_8}{q_{11}}\cdot\frac{q_{14}}{q_{13}}\cdot\frac{q_{2n}+q_{2n-2}+(-1)^{n-3\nu-2}(q_{6\nu+2}+q_{6\nu})}{20q_n^2}, \quad \text{by (6)}, \\
&\leqslant \frac{q_{11}+q_8}{q_{11}}\cdot\frac{q_{14}}{q_{13}}\cdot\frac{q_{2n}+q_{2n-2}+q_{2n-6}+q_{2n-8}}{20q_n^2} \\
&= \frac{q_{11}+q_8}{q_{11}}\cdot\frac{q_{14}}{q_{13}}\cdot\frac{q_{2n-4}}{2q_n^2}, \quad \text{by (7)}, \\
&= \frac{q_{11}+q_8}{q_{11}}\cdot\frac{q_{14}}{q_{13}}\cdot\frac{q_{n-2}^2+q_{n-3}^2}{2q_n^2}, \quad \text{by (3)}, \\
&\leqslant \frac{q_{11}+q_8}{q_{11}}\cdot\frac{q_{14}}{q_{13}}\left(\frac{q_6^2}{2q_8^2}+\frac{q_4^2}{2q_7^2}\right), \quad \text{by Lemma 1}, \\
&= 0{\cdot}20289\ldots < K.
\end{aligned}$$

LEMMA 4. *If $K \geqslant 21/100$, then the intersection of the boundaries of $H(0,0)$ and $H(1,0)$ in R_1 lies in or to the right of $H_1(1)$, if $n \geqslant 3$.*

Proof. The point of intersection is

$$(\tfrac{1}{2}\{1-\sqrt{(1-4K)}\},\ \tfrac{1}{2}\{1+\sqrt{(1-4K)}\})$$

and the required condition is

$$\left\{2-\frac{q_{n+1}}{q_n}-\tfrac{1}{2}(1-\sqrt{(1-4K)})\right\}\left\{2+\frac{q_{n-1}}{q_n}-\tfrac{1}{2}(1+\sqrt{(1-4K)})\right\} \leqslant K$$

since $H_1(1)$ is $H(2,-1)$, i.e.

$$\frac{1}{2}-\frac{q_{n+1}q_{n-1}}{q_n^2}+\tfrac{1}{2}\sqrt{(1-4K)}\left(\frac{q_{n+1}+q_{n-1}}{q_n}\right) \leqslant 0.$$

Now, for $n \geqslant 4$, $$\frac{q_{n+1}+q_{n-1}}{q_n} \leqslant \frac{q_6+q_4}{q_5} = \frac{9}{4}$$

and $$\frac{q_{n+1}q_{n-1}}{q_n^2} \geqslant \frac{q_5 q_3}{q_4^2} = \frac{24}{25}.$$

Hence $$\frac{1}{2}-\frac{q_{n+1}q_{n-1}}{q_n^2}+\tfrac{1}{2}\sqrt{(1-4K)}\left(\frac{q_{n+1}+q_{n-1}}{q_n}\right)$$

$$\leqslant \frac{1}{2}-\frac{24}{25}+\frac{1}{2}\cdot\frac{2}{5}\cdot\frac{9}{4} = -\frac{1}{100} < 0.$$

It may be verified that the result is true for $n = 3$ also.

LEMMA 5. *If $K \geqslant 21/100$, then adjacent members of the sequence $H_2(1),\ldots, H_2([\frac{1}{3}(n-2)])$ overlap without gaps in R_2.*

Proof. The required condition is

$$K \geqslant \frac{q_{n-3\nu-3}(q_{n+1}+q_{n+3\nu+1})(q_{n+1}+q_{n+3\nu+4})}{4q_n^2(q_{n+1}+q_{n+3\nu+3})} \quad \text{for } \nu = 1,\ldots, [\tfrac{1}{3}(n-5)].$$

This expression is less than

$$\frac{q_{n-3\nu-3}}{4q_n^2}\cdot\frac{q_{n+3\nu+4}}{q_{n+3\nu+3}}(q_{n+1}+q_{n+3\nu+1})$$

$$\leqslant \frac{q_{10}+q_{13}}{q_{13}}\cdot\frac{q_{16}}{q_{15}}\cdot\frac{1}{2}\left(\frac{q_6^2}{q_8^2}+\frac{q_6^2}{q_9^2}\right)$$

$$= 0{\cdot}20206\ldots < K.$$

LEMMA 6. *If $K \geqslant 21/100$, then the intersection of the boundaries of $H(0,0)$ and $H(0,1)$ in R_2 lies in or below $H_2(1)$ $(n \geqslant 4)$.*

Proof. The point of intersection is

$$\left(\left|\frac{q_{n+1}}{2q_n}+\frac{1}{2}\sqrt{\left(\frac{q_{n+1}^2}{q_n^2}-4K\frac{q_{n+1}}{q_{n-1}}\right)}\right|, \left\{-\frac{q_{n-1}}{2q_n}+\frac{1}{2}\sqrt{\left(\frac{q_{n-1}^2}{q_n^2}-4K\frac{q_{n-1}}{q_{n+1}}\right)}\right\}\right).$$

Hence the required condition is

$$\left\{1+\frac{3}{2}\frac{q_{n+1}}{q_n}-\frac{1}{2}\sqrt{\left(\frac{q_{n+1}^2}{q_n^2}-4K\frac{q_{n+1}}{q_{n-1}}\right)}\right\}\times$$

$$\times\left\{-1+\frac{3q_{n-1}}{2q_n}+\frac{1}{2}\sqrt{\left(\frac{q_{n-1}^2}{q_n^2}-4K\frac{q_{n-1}}{q_{n+1}}\right)}\right\}\leqslant K$$

since $H_2(1)$ is $H(1,2)$, i.e.

$$-\frac{5}{2}+2\frac{q_{n+1}q_{n-1}}{q_n^2}+\frac{q_{n+1}+q_{n-1}}{2q_n}\sqrt{\left(1-\frac{4K}{}\frac{q_n^2}{q_{n-1}q_{n+1}}\right)}\leqslant 0. \qquad (13)$$

Now, for $n \geqslant 4$,
$$\frac{q_{n+1}q_{n-1}}{q_n^2}\leqslant\frac{q_6q_4}{q_5^2}=\frac{65}{64},$$

and
$$\frac{q_{n+1}+q_{n-1}}{q_n}\leqslant\frac{q_6+q_4}{q_5}=\frac{9}{4}.$$

Hence the left-hand side of (13) is less than or equal to

$$-\frac{5}{2}+2\frac{65}{64}+\frac{9}{8}\sqrt{\left(1-4\frac{21}{100}\frac{64}{65}\right)}$$

$$=-0{\cdot}0009\ldots<0.$$

THEOREM 1. *For the form*

$$\left(x+\frac{q_{3m+1}}{q_{3m}}y\right)\left(x-\frac{q_{3m-1}}{q_{3m}}y\right)\quad(m\geqslant 1),$$

M_1 is $\frac{3}{4}q_{6m-3}/q_{3m}^2$, taken at $x\equiv\frac{1}{2}$, $y\equiv 0 \pmod 1$.

Proof. We take $n=3m$ in the definitions of $H(x,y)$ and R_2 and suppose that $m\geqslant 3$. The proof falls into two parts: we have first to show that

$$M(\tfrac{1}{2},0)=\frac{3q_{6m-3}}{4q_{3m}^2}$$

and secondly to show that regions $H(x,y)$, with

$$K=\frac{3q_{6m-3}}{4q_{3m}^2},$$

cover the plane. Now

$$\left(x-\frac{1}{2}+\frac{q_{3m+1}}{q_{3m}}y\right)\left(x-\frac{1}{2}-\frac{q_{3m-1}}{q_{3m}}y\right)=\frac{1}{4q_{3m}^2}(u-w)(v-w),$$

where $u=2(q_{3m}x+q_{3m+1}y)$, $v=2(q_{3m}x-q_{3m-1}y)$, $w=q_{3m}$. Since $q_{3m}^2-q_{3m-1}q_{3m+1}=(-1)^{3m}$, by (2), u is obtained when and only when

$$x=\tfrac{1}{2}(-1)^{3m}uq_{3m}-\lambda q_{3m+1}, \qquad y=\tfrac{1}{2}(-1)^{3m-1}uq_{3m-1}+\lambda q_{3m},$$

where λ (and later λ') is an arbitrary integer. Hence

$$\begin{aligned} v&=(-1)^{3m}u(q_{3m}^2+q_{3m-1}^2)-2\lambda(q_{3m}q_{3m+1}+q_{3m}q_{3m-1})\\ &=(-1)^{3m}q_{6m}u-2\lambda q_{6m+1}, \quad \text{by (3).}\end{aligned}$$

Hence, when $u=w+r$,

$$v=(-1)^{3m}q_{6m}r-2\lambda' q_{6m+1}+w, \quad \text{by (2),}$$

so that $\min|u-w||v-w|$ is $\min|r||rq_{6m}-2\lambda' q_{6m+1}|$. Now the convergents to the continued fraction for q_{6m}/q_{6m+1} are q_0/q_1, q_1/q_2, etc., and, since r is odd (because w is odd and u even) and $2\lambda'$ is even, the value of r following $q_{3\mu}$ which gives a smaller value for $|rq_{6m}-2\lambda' q_{6m+1}|$ is, by Lemma 2, $q_{3\mu+3}$ with $2\lambda'$ equal to $q_{3\mu+2}$. Hence

$$\begin{aligned} &\min|r||rq_{6m}-2\lambda' q_{6m+1}|\\ &\quad=\min_{0\leqslant\mu\leqslant 2m} q_{3\mu}|q_{3\mu}q_{6m}-q_{3\mu-1}q_{6m+1}|\\ &\quad=\min_{0\leqslant\mu\leqslant m} q_{3\mu}q_{6m-3\mu}\\ &\quad=\min \tfrac{1}{5}\{q_{6m+2}+q_{6m}+(-1)^{3\mu}(q_{6m-6\mu}+q_{6m-6\mu-2})\}\\ &\quad=q_3q_{6m-3}\\ &\quad=3q_{6m-3},\end{aligned}$$

which completes the first part of the proof. For the second part it is sufficient to show that a fundamental parallelogram of the lattice is covered. Now, if R_1 and R_2 are covered, then so, by symmetry about the points $(\frac{1}{2},\frac{1}{2})$ and $(q_{n+1}/2q_n, -q_{n-1}/2q_n)$, are the rectangles

$$\tfrac{1}{2}\leqslant\xi\leqslant 1, \qquad 0\leqslant\eta\leqslant 1$$

and $$0\leqslant\xi\leqslant q_{n+1}/q_n, \qquad -q_{n-1}/q_n\leqslant\eta\leqslant -q_{n-1}/2q_n.$$

These with R_1 and R_2 contain the triangle bounded by the lines $\xi=\eta$, $\xi=1-\eta$, and $q_{n-1}\xi+q_{n+1}\eta=0$, and, by symmetry of the lattice

about the lines $\xi = \pm\eta$, it follows that the covering of R_1 and R_2 is equivalent to the covering of the plane. Now

$$\begin{aligned}\frac{3q_{6m-3}}{4q_{3m}^2} &= \frac{3q_{3m-2}(q_{3m-1}+q_{3m-3})}{4q_{3m}^2}\\ &\geqslant \frac{3q_7}{4q_9}\left(\frac{q_9}{q_{10}}+\frac{q_7}{q_{10}}\right)\\ &= 0{\cdot}2445\ldots > \frac{21}{100},\end{aligned}$$

so that Lemmas 3, 4, 5, 6 apply.

Now $H_1(m-1)$ and $H(\frac{1}{2}+\frac{1}{2}(-1)^{m-1}q_{3m-3},\ \frac{1}{2}(-1)^{m-2}q_{3m-4})$, i.e. the reflection of $H_1(m-1)$ in the point $(\frac{1}{2},\frac{1}{2})$, cover between their centres so that, by Lemmas 3 and 4, R_1 is covered.

Also the regions $H_2(m-1)$ and $H_2(m)$ overlap without gaps in R_2 if

$$K > \frac{(q_{3m+1}+q_{6m-2})(q_{3m+1}+q_{6m+1})}{4q_{3m}^2(q_{3m+1}+q_{6m})}. \tag{14}$$

Now

$$\begin{aligned}\frac{(q_{3m+1}+q_{6m-2})(q_{3m+1}+q_{6m+1})}{(q_{3m+1}+q_{6m})} &< \frac{(q_{3m+1}+q_{6m-2})q_{6m+1}}{q_{6m}}\\ &< \omega(q_{3m+1}+q_{6m-2})\\ &\leqslant q_{6m-3}\,\omega\left(\frac{q_{10}}{q_{15}}+\frac{q_{16}}{q_{15}}\right)\\ &< 2{\cdot}764q_{6m-3}\\ &< 3q_{6m-3},\end{aligned}$$

so that the inequality in (14) is correct and, by Lemmas 5 and 6, R_2 is covered. This completes the proof of the theorem for $m \geqslant 3$. If $m = 1$ or 2, it can be verified that it is still correct, although the arguments used above are no longer valid. If $m = 0$, then $M_1(f) = \frac{1}{4}$, taken at $x \equiv \frac{1}{2}$, $y \equiv 0 \pmod 1$.

THEOREM 2. *For the form*

$$\left(x+\frac{q_{3m}}{q_{3m-1}}y\right)\left(x-\frac{q_{3m-2}}{q_{3m-1}}y\right),$$

$$M_1 = \frac{q_{6m-4}\,q_{6m-2}(q_{6m-1}-2)}{4q_{3m-1}^2(q_{6m-2}-1)^2}$$

taken at $x \equiv \frac{1}{2} \pm \frac{q_{6m-4}}{2q_{3m-1}(q_{6m-2}-1)}, \quad y \equiv 0 \pmod 1$

if m is even; or at

$$\left.\begin{aligned} x &\equiv \frac{1}{2} \pm \frac{q_{6m-4}}{2(q_{3m}+q_{3m-2})(q_{6m-2}-1)} \\ y &\equiv \mp \frac{q_{6m-4}}{(q_{3m}+q_{3m-2})(q_{6m-2}-1)} \end{aligned}\right\} \pmod 1$$

if m is odd.

Proof. Take $n = 3m-1$ and consider the cases $m \geqslant 2$. Just one of the points at which M_1 is taken lies in R_1; we denote it by P, and its coordinates are

$$\frac{1}{2} - \frac{q_{6m-4}}{2q_{3m-1}(q_{6m-2}-1)}, \qquad \frac{1}{2} - \frac{(-1)^m q_{6m-4}}{2q_{3m-1}(q_{6m-2}-1)}.$$

It has to be shown that P is not completely covered when

$$K = \frac{q_{6m-4}\,q_{6m-2}(q_{6m-1}-2)}{4q_{3m-1}^2(q_{6m-2}-1)^2}.$$

For every lattice point, $q_{3m-1}\xi$ and $q_{3m-1}\eta$ are integers, and

$$q_{3m-1}\eta \equiv (-1)^{m-1} q_{6m-2}\, q_{3m-1}\xi \pmod{q_{6m-1}}$$

so that, if $q_{3m-1}\xi = \frac{1}{2}q_{3m-1} + r$, then

$$q_{3m-1}\eta \equiv \tfrac{1}{2}q_{3m-1} + \tfrac{1}{2}q_{6m-1} + (-1)^{m-1} r q_{6m-2} \pmod{q_{6m-1}}.$$

Denote $\frac{1}{2}q_{6m-4}/(q_{6m-2}-1)$ by t; then $\frac{1}{2}\omega^{-2} < t < \frac{1}{4}$. It is required to show that

$$\min_{r,s} |r+t|\,|rq_{6m-2} - t - (s+\tfrac{1}{2})q_{6m-1}| \geqslant \frac{q_{6m-4}\,q_{6m-2}(q_{6m-1}-2)}{4(q_{6m-2}-1)^2}.$$

Since $|t| < \frac{1}{2}$, the sequence of least values of $|rq_{6m-2} - t - (s+\frac{1}{2})q_{6m-1}|$ as r takes the values $0, 1, \ldots$ or $-1, -2, \ldots$ is given by the same values of r as if t were 0, viz. $r = 0, 1, \ldots, \frac{1}{2}q_{3\mu-1}, \ldots$ or $-1, -4, \ldots, -\frac{1}{2}q_{3\mu-1}, \ldots$. Hence the minimum is the smaller of

$$\min_{0 \leqslant \mu \leqslant 2m} |\tfrac{1}{2}q_{3\mu-1} + t|\,|\tfrac{1}{2}(-1)^{3\mu-1} q_{6m-3\mu-1} - t|$$

and

$$\min_{1 \leqslant \mu \leqslant 2m} |\tfrac{1}{2}q_{3\mu-1} - t|\,|\tfrac{1}{2}(-1)^{3\mu-1} q_{6m-3\mu-1} + t|.$$

If we consider together the values μ and $2m-\mu$, this gives the smaller of

$$t(\tfrac{1}{2}q_{6m-1}-t) \quad \text{and} \quad \min_{1\leqslant\mu\leqslant m}\phi(\mu),$$

where $$\phi(\mu) = (\tfrac{1}{2}q_{3\mu-1}-t)\{\tfrac{1}{2}q_{6m-3\mu-1}+(-1)^{3\mu-1}t\}.$$

We have

$$\begin{aligned}
&\phi(\mu+1)-\phi(\mu)\\
&\quad = tq_{6m-3\mu-3}+(-1)^{3\mu}(\tfrac{1}{2}q_{6m-6\mu-4}+tq_{3\mu+1}-2t^2), \quad \text{by (6) and (7)},\\
&\quad > tq_{6m-3\mu-3}-\tfrac{1}{2}q_{6m-6\mu-4}-tq_{3\mu+1}-2t^2\\
&\quad > t(q_{6m-3\mu-3}-3q_{6m-6\mu-4}-q_{3\mu+1}-2t), \quad \text{since } t > \tfrac{1}{6},\\
&\quad \geqslant t(q_{6m-3\mu-3}-3q_{6m-3\mu-7}-q_{6m-3\mu-5}-2t)\\
&\quad = t(2q_{6m-3\mu-8}-2t)\\
&\quad > 0 \quad \text{for } \mu \leqslant m-1.
\end{aligned}$$

Hence

$$\begin{aligned}
\min_{1\leqslant\mu\leqslant m}\phi(\mu) = \phi(1) &= \frac{q_{6m-4}q_{6m-2}(q_{6m-1}-2)}{4(q_{6m-2}-1)^2}\\
&= t(\tfrac{1}{2}q_{6m-1}-t),
\end{aligned}$$

which completes the first part of the proof.

Now, for $m \geqslant 3$,

$$\begin{aligned}
\frac{q_{6m-4}q_{6m-2}(q_{6m-1}-2)}{4q_{3m-1}^2(q_{6m-2}-1)^2} &> \frac{q_{6m-4}(q_{6m-1}-2)}{4q_{3m-1}^2q_{6m-2}}\\
&\geqslant \frac{q_{17}-2}{4q_{16}}\cdot\frac{q_{3m-2}^2+q_{3m-3}^2}{q_{3m-1}^2}\\
&\geqslant \frac{q_{17}-2}{4q_{16}}\left(\frac{q_7^2}{q_8^2}+\frac{q_7^2}{q_9^2}\right)\\
&= 0{\cdot}2131\ldots,
\end{aligned}$$

while for $m = 2$ the value is $0{\cdot}2167\ldots$. Hence Lemmas 3, 4, 5, and 6 apply.

If m is even, then P is on the line $\xi = \eta$: the centres of $H_1(m-1)$ and $H_1(m)$, on whose boundaries P lies, are respectively above and below R_1. Now, since the centre of $H_1(m-1)$ is nearer the line $\eta = 0$ than is the centre of $H_1(m)$, the increase in the horizontal width of $H_1(m-1)$, as η increases from the value 0, is more rapid than is the decrease in the width of $H_1(m)$. Consequently the portion of R_1 above the line $\xi = \eta$ is covered and hence, by reflection in this line, so is the portion below. A similar argument applies when m is odd.

$H_2(m-1)$ and its reflection in $(\frac{1}{2}q_{3m}/q_{3m-1}, -\frac{1}{2}q_{3m-2}/q_{3m-1})$ completely cover between their centres if

$$K > \frac{q_{6m-3}}{4q_{3m-1}^2}.$$

Now
$$\frac{4q_{3m-1}^2 K}{q_{6m-3}} = \frac{q_{6m-4}q_{6m-2}(q_{6m-1}-2)}{q_{6m-3}(q_{6m-2}-1)^2}$$

and

$$\begin{aligned} &q_{6m-4}q_{6m-2}(q_{6m-1}-2)-q_{6m-3}(q_{6m-2}-1)^2 \\ &\quad = q_{6m-2}(q_{6m-4}q_{6m-1}-q_{6m-3}q_{6m-2})-2q_{6m-2}(q_{6m-4}-q_{6m-3})-q_{6m-3} \\ &\quad = q_{6m-2}-q_{6m-3}+2q_{6m-2}q_{6m-5} \\ &\quad > 0. \end{aligned}$$

Hence, from Lemmas 5 and 6, R_2 is completely covered and the theorem is proved for $m \geqslant 2$.

It may be verified that it is correct also if $m = 1$.

THEOREM 3. *Let A be the smaller root of the equation*

$$\psi(A) = 2q_{6m+2}A^2-(q_{6m-1}q_{6m+2}+2q_{6m+2}+q_{6m}+3)A+ \\ +(q_{6m-1}q_{6m}+2q_{6m}-q_{6m-4}) = 0.$$

Then, for the form

$$\left(x+\frac{q_{3m+2}}{q_{3m+1}}y\right)\left(x-\frac{q_{3m}}{q_{3m+1}}y\right) \quad (m \geqslant 1),$$

M_1 is $q_{6m+2}A(2-A)/4q_{3m+1}^2$; this is taken in R_2 at the point

$$P\left(\frac{q_{3m+2}+(-1)^m A}{2q_{3m+1}}, \frac{q_{6m+2}A-q_{6m}-q_{3m}}{2q_{3m+1}}\right).$$

Proof.

$$\begin{aligned} q_{6m+2}\psi(q_{6m}/q_{6m+2}) &= q_{6m}^2-q_{6m-4}q_{6m+2}-3q_{6m} \\ &= \tfrac{1}{5}(q_{12m+2}-q_{12m-2}-16-15q_{6m}), \quad \text{by (6)}, \\ &\geqslant \tfrac{1}{5}(q_{14}-q_{10}-16-15q_6), \quad \text{by Lemma 1}, \\ &= 62 > 0. \end{aligned}$$

Since the sum of the roots of $\psi(A) = 0$ is greater than 1, and

$$q_{6m}/q_{6m+2} \leqslant q_6/q_8 < \tfrac{1}{2}$$

we have
$$A > \frac{q_{6m}}{q_{6m+2}}. \qquad (15)$$

Also

$$\begin{aligned}\psi\left(\frac{q_{6m}+\frac{1}{2}}{q_{6m+2}}\right) &= \frac{1}{2}\left(\frac{2q_{6m}+1}{q_{6m+2}} - q_{6m-1} - 2\right) + \frac{(q_{6m}-3)(q_{6m}+\frac{1}{2})}{q_{6m+2}} - q_{6m-4} \\ &< \frac{q_{6m+1}}{q_{6m+2}} - \tfrac{1}{2}q_{6m-1} + \frac{q_{6m}^2}{q_{6m+2}} - q_{6m-4} \\ &< \omega^{-1} + \tfrac{13}{34}q_{6m} - \tfrac{1}{2}q_{6m-1} - q_{6m-4} \\ &\leqslant \omega^{-1} + 13(\tfrac{13}{34} - \tfrac{1}{2}\omega^{-1} - \tfrac{2}{13}) \\ &< 0.\end{aligned}$$

Hence
$$A < \frac{q_{6m}+\frac{1}{2}}{q_{6m+2}}. \tag{16}$$

By (15) and (16), P lies in R_2 as stated.

Since
$$q_{6m+2}A - q_{6m} = \frac{(q_{6m}-3)A - q_{6m-4}}{q_{6m-1}+2-2A},$$

we have, from (16),

$$\begin{aligned}q_{6m+2}A - q_{6m} &< \frac{(q_{6m}-3)(q_{6m}+\frac{1}{2}) - q_{6m-4}q_{6m+2}}{(q_{6m-1}+2)q_{6m+2} - 2q_{6m} - 1} \\ &< \frac{q_{6m}^2 - q_{6m-4}\,q_{6m+2} - \frac{5}{2}q_{6m}}{q_{6m-1}\,q_{6m+2}}.\end{aligned} \tag{17}$$

Now $4q_{3m+1}^2 M(a,b)$, evaluated at P, is the smaller of

$$\min|r-A|\,|-rq_{6m+2}+sq_{6m+3}-B| \quad (m \text{ even})$$

and
$$\min|r+A|\,|rq_{6m+2}-sq_{6m+3}-B| \quad (m \text{ odd}),$$

where $B = q_{6m+2}A - q_{6m}$, r is even and s is odd. Since r takes the values $0, \pm 2, \pm 4, \ldots$, the two minima are the same.

Since, by (15) and (16), $|A| < \frac{1}{2}$, $|B| < \frac{1}{2}$ we have, by reasoning similar to that used in the proof of Theorem 2, that the minimum is the least of

$$A(q_{6m+3}-B), \qquad \min_{1\leqslant\mu\leqslant 2m+1}\phi_1(\mu), \qquad \min_{1\leqslant\mu\leqslant 2m+1}\phi_2(\mu),$$

where
$$\phi_1(\mu) = (q_{3\mu-1}-A)\{q_{6m-3\mu+3}+(-1)^{3\mu-1}B\},$$
$$\phi_2(\mu) = (q_{3\mu-1}+A)\{q_{6m-3\mu+3}-(-1)^{3\mu-1}B\}.$$

For μ even, $\phi_2(\mu)$ is greater than $\phi_1(\mu)$ and need not be considered. By (6), (7), (8),

$$\phi_2(2\lambda+1) - \phi_2(2\lambda-1) = -8q_{6m-12\lambda+3} - 4Aq_{6m-6\lambda+3} - 4Bq_{6\lambda-1},$$

so that only $\phi_2(2m+1)$ need be considered. By (6),

$$\phi_1(\mu+1) - \phi_1(\mu) = 2Aq_{6m-3\mu+1} + 2(-1)^{3\mu}(q_{6m-6\mu} + Bq_{3\mu+1} - AB),$$

and, since this is positive for μ even, we may disregard $\phi_1(\mu)$ if μ is odd and greater than 1. By (8),

$$\phi_1(2\lambda+2)-\phi_1(2\lambda) = 4Aq_{6m-6\lambda}+8q_{6m-12\lambda-3}-4Bq_{6\lambda+2},$$

and, since this is a steadily decreasing function of λ, the minimum is one of $\phi_1(1)$, $\phi_1(2)$, $\phi_1(2m)$, $\phi_2(2m+1)$, and $A(q_{6m+3}-B)$.

Now
$$\begin{aligned}\phi_1(2)-\phi_1(1) &= 2Aq_{6m-2}-2q_{6m-6}-10B+2AB\\ &> 2Aq_{6m-2}-2q_{6m-6}-5\\ &\geqslant 2Aq_{10}-2q_6-5, \quad \text{if } m \geqslant 2,\\ &> 2\omega^{-2}q_{10}-2q_6-5\\ &> 0.\end{aligned}$$

Also

$$\begin{aligned}&\phi_2(2m+1)-\phi_1(1)\\ &= (q_{6m+2}-2q_{6m})+A(q_{6m}+1)-(q_{6m+2}+2)B\\ &> q_{6m-1}+\frac{q_{6m}^2}{q_{6m+2}}-\frac{q_{6m}^2-q_{6m-4}q_{6m+2}-\frac{5}{2}q_{6m}}{q_{6m-1}}-1, \quad \text{by (15), (16), (17),}\\ &> \frac{U}{q_{6m+2}q_{6m-1}}-1+\tfrac{5}{2}\omega,\end{aligned}$$

where

$$\begin{aligned}U &= q_{6m-1}^2q_{6m+2}+q_{6m}^2q_{6m-1}-q_{6m}^2q_{6m+2}+q_{6m-4}q_{6m+2}^2\\ &= q_{6m-1}^2q_{6m+2}+q_{6m-4}q_{6m+2}^2-2q_{6m}^3, \quad \text{by (5),}\\ &= q_{6m-1}^2q_{6m+2}+q_{6m-4}q_{6m+2}^2-2q_{6m}(q_{6m-2}q_{6m+2}-1), \quad \text{by (2),}\\ &> q_{6m+2}(q_{6m-1}^2-q_{6m-2}q_{6m}+q_{6m-4}q_{6m+2}-q_{6m}q_{6m-2})\\ &= 2q_{6m+2} > 0,\end{aligned}$$

so that $\phi_2(2m+1) > \phi_1(1)$. Finally,

$$\begin{aligned}\phi_1(2m) &= (q_{6m-1}-A)(q_{6m}+3-q_{6m+2}A)\\ &= q_{6m+2}A^2-A(q_{6m}+3+q_{6m-1}q_{6m+2})+q_{6m-1}q_{6m}+2q_{6m}-q_{6m-4}\\ &= q_{6m+2}A(2-A)+\psi(A)\\ &= \phi_1(1), \quad \text{since } \psi(A) = 0,\\ &= A(q_{6m+3}-B),\end{aligned}$$

which completes the first part of the proof.

Now take
$$K = \frac{q_{6m+2}A(2-A)}{4q_{3m+1}^2}.$$

Since $1 > A > \omega^{-2}$, we have

$$\begin{aligned} K &> q_{6m+2}\,\omega^{-2}(2-\omega^{-2})/4q_{3m+1}^2 \\ &= \tfrac{1}{4}\omega^{-2}(2-\omega^{-2})\,\frac{q_{3m+1}^2+q_{3m}^2}{q_{3m+1}^2}, \text{ by (3)}, \\ &\geqslant \tfrac{1}{4}\omega^{-2}(2-\omega^{-2})\,q_8/q_4^2 \\ &> \tfrac{21}{100}, \end{aligned}$$

so that Lemmas 3, 4, 5, 6 apply.

Also $H_1(m)$ and its reflection in $(\frac{1}{2}, \frac{1}{2})$ completely cover between their centres if

$$K > \tfrac{1}{4}q_{6m+1}/q_{3m+1}^2.$$

Now $$q_{6m+2}\,\omega^{-2}(2-\omega^{-2})/q_{6m+1} = q_{6m+2}/\omega q_{6m+1} > 1,$$

so that R_1 is covered completely.

Let R_3 denote the rectangle

$$-q_{3m} \leqslant q_{3m+1}\,\eta \leqslant 0 \text{ with } 0 \leqslant 2q_{3m+1}\,\xi \leqslant q_{3m+2} \quad \text{if } m \text{ is odd},$$

and $$q_{3m+2} \leqslant 2q_{3m+1}\,\xi \leqslant 2q_{3m+2} \quad \text{if } m \text{ is even}.$$

Let $H_3(\mu)$ denote the region

$$H\big(\tfrac{1}{2}(-1)^{m-\mu}q_{3\mu+2}, \tfrac{1}{2}-\tfrac{1}{2}(-1)^{m-\mu}q_{3\mu+1}\big),$$

whose centre, by (1), (2), (3), is at the point

$$\left(\frac{q_{3m+2}+(-1)^m q_{3m-3\mu-1}}{2q_{3m+1}},\ \frac{-q_{3m}+(-1)^{m-\mu}q_{3m+3\mu+3}}{2q_{3m+1}}\right).$$

I shall show that each adjacent pair of the sequence

$$H_3(0),\ H_3(1),\ldots,\ H_3(m-1),$$

overlap without gaps in R_3. This is so if

$$K > \frac{q_{3m-3\mu}(q_{3m}+q_{3m+3\mu})(q_{3m}+q_{3m+3\mu+3})}{4q_{3m+1}^2(q_{3m}+q_{3m+3\mu+2})} \quad \text{for } \mu = 1,\ldots,\ m-1.$$

But this expression is less than

$$\frac{q_6+q_9}{q_9}\cdot\frac{q_{12}}{q_{11}}\cdot\frac{q_{14}+q_{12}+q_6+q_4}{20q_7^2} = 0{\cdot}1952\ldots,$$

and $K > 0{\cdot}21$.

Now the boundaries of $H_2(m-1)$, $H_3(m-1)$, and $H_3(m)$ meet in P. Since the centre of $H_3(m-1)$ is nearer to P than is the centre of $H_3(m)$, $H_3(m)$ and $H_3(m-1)$ cover the part of R_3 below the level of P. By Lemmas 5 and 6, the part above it is covered. Hence R_3 is covered and so is its reflection in $(\frac{1}{2}q_{3m+2}/q_{3m+1}, -\frac{1}{2}q_{3m}/q_{3m+1})$. Thus R_2 is covered and this completes the proof of the theorem.

The use of Lemmas 2, 4, 6 restricts the proof to the cases $m \geqslant 1$.

When $m = 0$, the theorem is false and in this case $M_1 = \frac{4}{9}$ taken at $x \equiv \pm\frac{2}{9}$, $y \equiv \pm\frac{4}{9} \pmod 1$.

4. For approximations to the form x^2-2y^2 there is only one case to be considered and we have

THEOREM 4. *For the form* $x^2-p_n^2q_n^{-2}y^2$, M_1 *is* $\frac{1}{4}p_{2n}q_n^{-2}$, *taken at* $x \equiv 0$, $y \equiv \frac{1}{2} \pmod 1$.

Proof.
$$\left|x^2-\frac{p_n^2}{q_n^2}(y-\tfrac{1}{2})^2\right| = \frac{1}{4q_n^2}|u-p_n||v-p_n|,$$

where
$$u = -2q_nx+2p_ny, \qquad v = 2q_nx+2p_ny.$$

Since, by (9) and (2),
$$-2q_np_{n-1}+2p_nq_{n-1} = 2(-1)^{n+1},$$

we have
$$\begin{aligned} v &= 2q_n\{(-1)^{n+1}\tfrac{1}{2}up_{n-1}+\lambda p_n\}+2p_n\{(-1)^{n+1}\tfrac{1}{2}uq_{n-1}+\lambda q_n\} \\ &= (-1)^{n+1}uq_{2n}+2\lambda q_{2n+1}, \end{aligned}$$

by (9) and (3), where λ is any integer.

Hence, if $u = p_n+r$, then
$$v = p_n+(-1)^{n+1}rq_{2n}+(2\lambda+1)q_{2n+1}$$

and
$$\min|u-p_n||v-p_n| = \min|r||rq_{2n}-(2\lambda+1)q_{2n+1}|.$$

Since p_n is odd and u is even, r is odd.

Now the convergents to q_{2n}/q_{2n+1} are $q_0/q_1, q_1/q_2, \ldots$ and, from Lemma 2, in the sequence of minima of $|rq_{2n}-sq_{2n+1}|$, the term $|q_{\mu-1}q_{2n}-q_{\mu-2}q_{2n+1}|$ is followed by

$$|q_\mu q_{2n}-q_{\mu-1}q_{2n+1}|, \quad |p_\mu q_{2n}-p_{\mu-1}q_{2n+1}|, \quad |2q_\mu q_{2n}-2q_{\mu-1}q_{2n+1}|,$$

and
$$|q_{\mu+1}q_{2n}-q_\mu q_{2n+1}|.$$

Since we must choose r and s odd, and since q_μ is even when $q_{\mu+1}$ is odd and vice versa, the required minimum is

$$\begin{aligned} &\min p_\mu|p_\mu q_{2n}-p_{\mu-1}q_{2n+1}| \\ &\quad = \min_{0\le\mu\le n} p_\mu p_{2n-\mu}, \quad \text{by (9) and (2)}, \\ &\quad = \min_{0\le\mu\le n}\{p_n^2+2(-1)^{\mu-1}q_{n-\mu-1}\}, \quad \text{by (10)}, \\ &\quad = p_0p_{2n} \\ &\quad = p_{2n}. \end{aligned}$$

This completes the first part of the proof.

As a fundamental region which must be shown to be covered we take the fundamental parallelogram of the lattice for which $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. Because of the symmetry of the lattice and hyperbolas about the lines $x = \frac{1}{2}, y = \frac{1}{2}$, it is enough to cover $0 \leqslant x \leqslant \frac{1}{2}, 0 \leqslant y \leqslant \frac{1}{2}$. We shall in fact cover the larger region

$$S\ (0 \leqslant \xi \leqslant \tfrac{1}{2}+\tfrac{1}{2}p_n/q_n,\ -\xi \leqslant \eta \leqslant 1-\xi).$$

Let $H(x_0, y_0)$ denote the region

$$|(x-x_0)^2-(p_n^2/q_n^2)(y-y_0)^2| \leqslant K.$$

Let $H_1(\nu)$ denote the region $H\big((-1)^{\nu-1}q_\nu, \tfrac{1}{2}+\tfrac{1}{2}(-1)^\nu p_\nu\big)$ with centre at

$$\left(\frac{p_n-p_{n-\nu-1}}{2q_n},\ \frac{-p_n+(-1)^{\nu-1}p_{n+\nu+1}}{2q_n}\right),$$

by (11) and (12), and $H_2(\nu)$ denote the region $H\big((-1)^\nu q_\nu, \tfrac{1}{2}+\tfrac{1}{2}(-1)^{\nu-1}p_\nu\big)$ with centre at

$$\left(\frac{p_n+p_{n-\nu-1}}{2q_n},\ \frac{-p_n+(-1)^\nu p_{n+\nu+1}}{2q_n}\right).$$

I show that the regions $H_1(-1)$, $H_1(0)$,..., $H_1(n-1)$, $H_2(n-1)$,..., $H_2(0)$ cover S. For each of $H_1(0)$,..., $H_2(0)$ the centre satisfies $|\xi+\eta| \geqslant 2$ and so lies outside S. Also the centres lie alternately above and below S.

Now $H_1(\nu)$, $H_1(\nu+1)$ completely cover between their centres if

$$K > \frac{(p_{n+\nu+2}+p_{n+\nu+1})(p_{n-\nu-1}-p_{n-\nu-2})}{16q_n^2}$$

$$= \frac{q_{n+\nu+2}q_{n-\nu-2}}{4q_n^2}$$

$$= \frac{p_{2n+1}+(-1)^{n-\nu-2}p_{2\nu+3}}{16q_n^2}, \quad \text{by (11) and (12).}$$

For $\nu = 0, 1,..., n-2$ this expression is less than or equal to

$$\frac{p_{2n+1}+p_{2n-1}}{16q_n^2} = \frac{q_{2n}}{4q_n^2} < \frac{p_{2n}}{4q_n^2} = K.$$

$H_1(n-1)$ and $H_2(n-1)$ cover between their centres if

$$K \geqslant \frac{p_{2n}}{4q_n^2},$$

which is the case.

$H_2(\nu+1)$ and $H_2(\nu)$ completely cover between their centres if

$$K > \frac{(p_{n+\nu+2}+p_{n+\nu+1})(p_{n-\nu-1}-p_{n-\nu-2})}{16q_n^2},$$

which has been verified above to be correct for $\nu = 0,\ldots, n-2$. Thus the portion of S for which $p_n - p_{n-1} \leqslant 2q_n\xi \leqslant 2q_n$ is proved to be covered.

Now $H_1(-1)$ covers the portion of S for which $0 \leqslant \xi \leqslant \sqrt{K}$, and

$$\sqrt{K} = \sqrt{p_{2n}/2q_n} = \sqrt{(p_n^2 - 2q_{n-1}^2)/2q_n}, \quad \text{by (10).}$$

Now

$$\begin{aligned} p_n^2 - 2q_{n-1}^2 - (p_n - p_{n-1})^2 &= 2p_n p_{n-1} - p_{n-1}^2 - 2q_{n-1}^2 \\ &= 4q_{n-1}p_{n-1} + (-1)^n \\ &> 0, \end{aligned}$$

so that

$$\sqrt{K} > (p_n - p_{n-1})/2q_n.$$

$H_2(0)$ covers the portion of S for which $1 \leqslant \xi \leqslant \frac{1}{2} + \frac{1}{2}p_n/q_n$ if the point $\frac{1}{2} + \frac{1}{2}p_n/q_n, \frac{1}{2} - \frac{1}{2}p_n/q_n$ lies in $H_2(0)$, since the line $\xi = \eta + p_n/q_n$ either meets the lower right-hand quarter of $H_2(0)$ in two points symmetrical about the line $\xi + \eta = 2$ or not at all (in which latter case there is nothing to prove).

But $(\frac{1}{2} + \frac{1}{2}p_n/q_n, \frac{1}{2} - \frac{1}{2}p_n/q_n)$ lies in $H_2(0)$ if

$$K \geqslant \left(-\frac{1}{2} + \frac{p_n}{2q_n}\right)\left(\frac{1}{2} + \frac{p_n}{2q_n}\right) = \frac{p_n^2 - q_n^2}{4q_n^2}.$$

Now $p_n^2 - q_n^2 < p_n^2 - 2q_{n-1}^2 = p_{2n}$, and this completes the proof of the theorem.

5. The limits to which $M_1(f)$, and the values of a and b for which they are taken, tend as m tends to infinity in Theorems 1, 2, 3 or as n tends to infinity in Theorem 4, are as follows.

In Theorem 1, M_1 tends to $3\sqrt{5}/4\omega^4 = 0{\cdot}2446\ldots$, a equals $\frac{1}{2}$ and b equals 0.

In Theorem 2, M_1 tends to $\sqrt{5}/4\omega^2 = 0{\cdot}2135\ldots$, a tends to $\frac{1}{2}$ and b tends to 0.

In Theorem 3, M_1 tends to $\sqrt{5}/4\omega^2 = 0{\cdot}2135\ldots$, a tends to 0 and b tends to $\frac{1}{2}$.

Davenport (**3**) shows that, for the limit form $x^2 + xy - y^2$, M_1 is $\frac{1}{4}$ taken at $a = \frac{1}{2}, b = 0$, or $a = 0, b = \frac{1}{2}$, or $a = \frac{1}{2}, b = \frac{1}{2}$, while M_2 is $\frac{1}{5}$.

In Theorem 4, M_1 tends to $\sqrt{2} - 1 = 0{\cdot}4142\ldots$, a equals 0 and b equals $\frac{1}{2}$.

Varnavides (**4**) shows that, for the limit form $x^2 - 2y^2$, M_1 is $\frac{1}{2}$, taken at $a = 0, b = \frac{1}{2}$, while M_2 is $\frac{1}{4}$. He also shows in (**5**) that the form $x^2 - 2y^2$ is exceptional in the sense that for any form whose coefficients are only slightly different, M_1 (suitably scaled for comparison) is less than $12/25$.

This is a special quantitative case of Rogers's general qualitative result mentioned in § 1 above. For forms of the type considered by Varnavides which occur in Theorem 4, $M_1 \leqslant 239/576 = 0{\cdot}41493\ldots$.

6. Theorem K of Barnes and Swinnerton-Dyer (**1**) enables the proof of Theorem 4 to be shortened somewhat, but fails to give the results of Theorems 1, 2, 3.

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[*Added 23 January 1954*]

At the suggestion of the referee I have evaluated the inhomogeneous minima of the forms

$$f_n = \{x+(p_n/q_n)y\}\{x+(1-p_n/q_n)y\},$$

where p_n/q_n is a convergent to $\frac{1}{2}(\sqrt{13}+1) = [2,3,3,3,\ldots]$.

As $n \to \infty$, $\qquad f_n \to f = x^2+xy-3y^2$,

and this form is of special interest since it is the only known norm-form for which the first minimum is taken at an infinity of points in each fundamental region. The result is that

$$M_1(f_n) \to \tfrac{1}{2}(11\sqrt{13}-39) = 0{\cdot}3305\ldots < \tfrac{1}{3} = M_1(f),$$

so that the inequality

$$\limsup M_1(f_n) < M_1(f)$$

holds in this case also.

The proof is similar to that of Theorem 3 above.

INTEGRAL TRANSFORMS AND EIGENFUNCTION THEORY

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1. LET L^2 denote the space of functions f for which

$$\|f\|^2 = \int_0^\infty |f(x)|^2\,dx < \infty.$$

Let $k(t)$ be a non-decreasing function defined for all t, continuous at $t = 0$, with $k(0) = 0$, and denote by $\mathscr{L}^2$ the space of functions F for which

$$\|F\|^2 = \int_{-\infty}^\infty |F(t)|^2\,dk(t) < \infty,$$

this being a Lebesgue–Stieltjes integral. (All intervals involved in the Lebesgue–Stieltjes integrals are supposed open on the left and closed on the right.) Then $\mathscr{L}^2$, as well as L^2, is a Banach space with inner product [(**2**), § 42]. Let $\phi(x, w)$ be real, and continuous over any finite rectangle with $x \geqslant 0$, and define the kernels ρ, χ by the relations

$$\rho(x,t) = \int_0^x \phi(y,t)\,dy, \tag{1.1}$$

$$\chi(x,w) = \int_0^w \phi(x,t)\,dk(t). \tag{1.2}$$

Then the equations

$$\int_0^w F(t)\,dk(t) = \int_0^\infty f(x)\,\chi(x,w)\,dx, \tag{1.3}$$

$$\int_0^x f(y)\,dy = \int_{-\infty}^\infty F(t)\,\rho(x,t)\,dk(t), \tag{1.4}$$

under appropriate conditions, will define a transformation $Tf = F$ from L^2 to $\mathscr{L}^2$, and its inverse. Conditions on ρ, χ will be obtained which ensure that T is unitary.

This problem is of a familiar type in the theory of bounded integral transforms. When

$$k(t) = \begin{cases} t & (t > 0), \\ 0 & (t < 0), \end{cases}$$

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it reduces to a case solved by Bochner [(**1**), Theorems 77, 78], from which may be derived very simply a number of special results such as Plancherel's theorem for Fourier transforms. Again, it is well known that a function $k(t)$, and so a space $\mathscr{L}^2$, may be associated with a second-order linear differential equation, subject to suitable boundary conditions, in the singular case. This suggests that the Parseval relation and the expansion theorem for such an equation may be derived from the general theory of (1.3), (1.4). It is shown below that this may be done quite easily. A number of transforms in the sense of (1.3), (1.4) are given, and finally these are applied to derive a theorem of Hartman and Wintner on the spectrum.

2. It is obvious that the following general theorem may be obtained by trivial changes in the proof given by Bochner. In fact, as remarked by Bochner, further generalizations are easily derived by his methods. It is convenient to use the abbreviation $k(w, w')$ to denote $k(W+0)$, if $W = \min(w, w') > 0$, zero if w, w' have opposite sign, and $-k(W'+0)$ when $W' = \max(w, w') < 0$.

THEOREM 1. *The following conditions are necessary and sufficient for the transformation $Tf = F$, as defined by* (1.3), (1.4), *to be unitary from L^2 to $\mathscr{L}^2$:*

$$\int_{-\infty}^{\infty} \rho(x,t)\,\rho(y,t)\,dk(t) = \min(x,y) \quad (\textit{all } x, y \geqslant 0), \tag{2.1}$$

$$\int_{0}^{\infty} \chi(x,w)\,\chi(x,w')\,dx = k(w,w'), \tag{2.2}$$

at all points of continuity of $k(t)$.

Suppose that these conditions are satisfied. If $f \in L^2$, and

$$F_a(t) = \int_0^a f(x)\,\phi(x,t)\,dt, \tag{2.3}$$

then $F_a \to F = Tf$ in $\mathscr{L}^2$ norm as $a \to \infty$. If $F \in \mathscr{L}^2$, and

$$f_{ab}(x) = \int_a^b F(t)\,\phi(x,t)\,dk(t), \tag{2.4}$$

then $f_{ab} \to f = T^{-1}F$ in L^2 norm as $a \to -\infty$, $b \to \infty$.

3. Consider now the differential equation

$$\frac{d^2y}{dx^2} + \{\lambda - q(x)\}y = 0, \tag{3.1}$$

where $q(x)$ is a real function, continuous for $x \geqslant 0$, λ a complex parameter, and $\phi(x) = \phi(x,\lambda)$, $\theta(x) = \theta(x,\lambda)$ those solutions which satisfy, for all λ, the real boundary conditions

$$\phi(0) = \sin\alpha, \qquad \phi'(0) = -\cos\alpha, \qquad \theta(0) = \cos\alpha, \qquad \theta'(0) = \sin\alpha.$$

It will be clear that other forms of the differential equation may be treated in a similar manner, for the half-line $x \geqslant 0$. The basic Sturm–Liouville theory and Weyl's theory of the limit-point and the limit-circle cases will be assumed, as given by Titchmarsh [(**7**), §§ 1.5–2.2]. Following Levinson (**4**), (**5**), use is made of Helly's selection theorem, the Helly–Bray theorem, and related results [(**8**), 26–33]. Otherwise, §§ 4, 5 are self-contained. Subsequently some standard properties of (3.1), available in (**7**), are used.

From Weyl's theory we need the following results. Let $\psi_b(x,\lambda)$ be that solution of (3.1) which satisfies, for given $\lambda = u+iv$ $(v \neq 0)$, the boundary condition

$$\psi_b(b,\lambda)\cos\beta + \psi_b'(b,\lambda)\sin\beta = 0$$

with β real, and define $l(b,\lambda)$ by the equation

$$\psi_b(x,\lambda) = \theta(x,\lambda) + l(b,\lambda)\,\phi(x,\lambda).$$

For b, λ fixed, as β varies, $l(b,\lambda)$ describes, in the complex plane, a circle C_b with radius

$$R(b,\lambda) = \left\{2v \int_0^b |\phi(x,\lambda)|^2\,dx\right\}^{-1}. \tag{3.2}$$

When $b \to \infty$, the circles C_b converge, if $v \neq 0$, to a limit point or a limit circle. In either case, as $b \to \infty$,

$$|l(b,\lambda)| < K(\lambda) \tag{3.3}$$

if λ is not real, and, for given b, $l(b,\lambda)$ is a meromorphic function with poles on the real λ-axis. Hence, in the limit-point case,

$$m(\lambda) = \lim_{b\to\infty} l(b,\lambda)$$

exists as an analytic function, regular in either half-plane. Titchmarsh [(**7**), §§ 2.2, 3.1] uses this function also in the limit-circle case, and thus requires no distinction in his theorems between the two possibilities. This seems to require some further discussion. Levinson appears to abandon the limit-circle case in his second paper [e.g. (**14**) of (**5**)]. While the function $k(t)$ is not unique in this case, the whole theory may in fact be put in a form in which the type of the equation is immaterial, except for a few details. This will be done by a modification of the approach used by Levinson, making necessary a review of his preliminaries.

4. By the Sturm–Liouville theory for the interval $(0, b)$, there exists an increasing sequence of eigenvalues λ_{nb}, the poles of $l(b, \lambda)$, and an orthonormal set of eigenfunctions $r_{nb}^{\frac{1}{2}} \phi(x, \lambda_{nb})$. Denote by $k_b(t)$ the lower semicontinuous step-function which increases by r_{nb} as t passes through λ_{nb}, is otherwise constant, and $k_b(0) = 0$. By (3.1), the boundary conditions for $(0, b)$, and the Parseval formula,

$$\int_0^b \psi_b(x, \lambda)\psi_b(x, \mu)\, dx = \int_{-\infty}^{\infty} \frac{dk_b(t)}{(\lambda - t)(\mu - t)} = \frac{l(b, \lambda) - l(b, \mu)}{\mu - \lambda} \tag{4.1}$$

if λ, μ are not real. In particular, by (3.3), we obtain the results

$$\int_{-\infty}^{\infty} \frac{dk_b(t)}{1+t^2} < K, \qquad |k_b(t)| < K(1+t^2), \tag{4.2}$$

due to Titchmarsh. By the theorems of (**8**) previously quoted, we may find a sequence of values of b, tending to infinity, for which $k_b(t)$ tends to a limit function $k(t)$, with the properties listed in § 1, and such that

$$\int_{-\infty}^{\infty} \frac{dk(t)}{1+t^2} \leqslant K, \qquad |k(t)| \leqslant K(1+t^2). \tag{4.3}$$

By (4.2), (3.3) we may select a subsequence of this sequence of values such that, as $b \to \infty$ through the subsequence,

$$\int_{-\infty}^{\infty} \frac{dk_b(t)}{1+t^2} \to a_1, \qquad l(b, i) \to a_2, \tag{4.4}$$

for some constants a_1, a_2. *In the sequel, '$b \to \infty$' will mean through this sequence.*

Suppose now that $\zeta(t)$ is any continuous function which tends to zero as $|t| \to \infty$. Then, as $b \to \infty$,

$$\int_{-\infty}^{\infty} \frac{\zeta(t)\, dk_b(t)}{1+t^2} \to \int_{-\infty}^{\infty} \frac{\zeta(t)\, dk(t)}{1+t^2}. \tag{4.5}$$

For we may fix A so large that the integrals over $(-\infty, -A)$, (A, ∞) are dominated by ϵK, using (4.2), and then apply the Helly–Bray theorem

to the integral over $(-A, A)$. From (4.1) with $\mu = i$ and $v > 0$, we have

$$l(b,\lambda) = l(b,i)+(i-\lambda)\int_{-\infty}^{\infty}\left\{\frac{1}{(\lambda-t)(i-t)}-\frac{1}{1+t^2}\right\}dk_b(t)+$$

$$+(i-\lambda)\int_{-\infty}^{\infty}\frac{dk_b(t)}{1+t^2}.$$

It now follows from (4.4), (4.5) that, as $b \to \infty$, $l(b,\lambda)$ tends to a limit function $m(\lambda)$ (say), satisfying

$$m(\lambda) = a_2+(i-\lambda)\int_{-\infty}^{\infty}\frac{dk(t)}{(\lambda-t)(i-t)}+(i-\lambda)a_3,$$

where

$$a_3 = a_1-\int_{-\infty}^{\infty}\frac{dk(t)}{1+t^2},$$

provided that $v > 0$. If also $\mathrm{im}(\mu) > 0$, then

$$\frac{m(\lambda)-m(\mu)}{\mu-\lambda} = \int_{-\infty}^{\infty}\frac{dk(t)}{(\lambda-t)(\mu-t)}+a_3. \tag{4.6}$$

Clearly $m(\lambda)$ is an analytic function of λ, regular in the half-plane $v > 0$, and $l(b,\lambda) \to m(\lambda)$ uniformly in any bounded region in which $v > \delta > 0$.

If

$$\psi(x,\lambda) = \theta(x,\lambda)+m(\lambda)\phi(x,\lambda),$$

then, by (3.3), (4.1) and Fatou's theorem, $\psi(x,\lambda)$ is an L^2 solution of (3.1) when $v > 0$.

Similarly, we may define a function $m_1(\lambda)$ with the above properties in the half-plane $v < 0$, but $m(\lambda)$, $m_1(\lambda)$ are not in general analytic continuations of one another. It will therefore be convenient to restrict our attention to the upper half-plane. Note, however, that (4.6) holds also with $\mu = \bar{\lambda}$, if $m(\mu)$ is interpreted as the conjugate of $m(\lambda)$, since we may repeat the argument, beginning with (4.1) with $\mu = \bar{\lambda}$. This convention will be used frequently. It is shown later that the constant a_3 in (4.6) is zero.

Now take $\mu = \bar{\lambda}$ in (4.6), integrate with respect to u over (u_1, u_2), and make $v \to 0$. This is justified in § 3 of (**4**) by the procedure used below for (5.2). It follows that

$$-\frac{1}{\pi}\lim_{v\to 0}\int_{u_1}^{u_2}\mathrm{im}\{m(u+iv)\}\,du = k(u_2)-k(u_1) \tag{4.7}$$

holds, if u_1, u_2 are points of continuity of $k(t)$, in both the limit-point and the limit-circle case. Also the function $k(t)$ has been shown to be unique, except for an additive constant, in the limit-point case. This brings the theory into agreement with Lemma 3.3 of (7), apart from the constant factor due to differences of definition.

We have now defined the functions $m(\lambda)$, $\psi(x,\lambda)$, $k(t)$. The notation of § 1 will be supposed to refer to these functions, and to the solution $\phi(x,t)$ of (3.1). It may happen, however, that $k_b(0) = k(0) = 0$ does not hold, but the difficulty may be avoided by a suitable change of the λ-origin and $q(x)$. It is now necessary to verify (2.1) and (2.2).

5. These results may be proved in various ways. The shortest method for (2.1) appears to be to prove first of all the Parseval relation for functions of L^2 which vanish for $x > a$. The following is taken from Levinson's paper (4). Let

$$L\{f(x)\} = q(x)f(x) - f''(x), \tag{5.1}$$

and suppose that $f(x)$ has a continuous second derivative, but vanishes in the neighbourhood of $x = 0$ and for $x > a$. With the notation of (2.3), we have

$$\int_0^\infty \phi(x,t)\, L\{f(x)\}\, dx = t\, F_a(t),$$

these being continuous functions of t. Applying the Parseval relation for $(0,b)$,

$$\int_R^\infty |F_a(t)|^2\, dk_b(t) \leqslant \frac{1}{R^2} \int_R^\infty t^2 |F_a(t)|^2\, dk_b(t)$$

$$\leqslant \frac{1}{R^2} \|L\{f\}\|^2,$$

and similarly for the integral over $(-\infty, -R)$. Making $b \to \infty$ in the relation

$$\int_0^b |f(x)|^2\, dx = \int_{-\infty}^\infty |F_a(t)|^2\, dk_b(t),$$

and using the Helly–Bray theorem for the integral over $(-R, R)$, we see that $\|f\| = \|F_a\|$. Then (2.1) may be deduced by the usual procedure for integral transforms.

In proving (2.2), it will be convenient to use the notation

$$\|f(x,y)\|_R = \left\{ \int_0^R |f(x,y)|^2\, dx \right\}^{\frac{1}{2}}$$

(where x is always the variable of integration), and to collect together some preliminary results in the form of a lemma, for the functions

$$\chi_b(x,w) = \int_0^w \phi(x,t)\,dk_b(t),$$

$$\sigma_b(x,\lambda) = -\frac{1}{\pi}\,\mathrm{im}\int_0^u \psi_b(x,w+iv)\,dw,$$

$$\sigma(x,\lambda) = -\frac{1}{\pi}\,\mathrm{im}\int_0^u \psi(x,w+iv)\,dw \quad (\lambda = u+iv,\ v>0).$$

LEMMA.

(i) $\|\chi_b(x,w)\|_b < K(w)$, $\chi(x,w)\in L^2$, *and* $\|\chi_b(x,w)-\chi(x,w)\|_R \to 0$ *as* $b\to\infty$ *with* R *fixed.*

(ii) $\|\psi_b(x,\lambda)\|_b < K(u)/v$, *and* $\|\psi_b(x,\lambda)-\psi(x,\lambda)\|_b \to 0$ *as* $b\to\infty$, $v>0$.

(iii) $\sigma(x,\lambda)\in L^2$ *for* $v>0$, *and* $\|\sigma(x,\lambda)\| < K(u)$ *as* $v\to 0$.

(iv) $\|\sigma(x,\lambda)-\chi(x,u)\|\to 0$ *as* $v\to 0$, *at all points* u *of continuity of* $k(t)$.

With the notation of (2.2), we have, from the Sturm–Liouville theory, or by Theorem 1,

$$\int_0^b \chi_b(x,w)\,\chi_b(x,w')\,dx = k_b(w,w').$$

The first part of the lemma follows from this result and (4.2). The second part follows from (3.2), (3.3), (4.1), since

$$|l(b,\lambda)-m(\lambda)| \leqslant R(b,\lambda).$$

Also (iii) will be obvious from the following proof of (iv).

Let $r_{nb}^{\frac{1}{2}}s(u,\lambda_{nb})$ denote the 'Fourier coefficient' of $\chi_b(x,u)$. By the Parseval relation, we have

$$\|\sigma_b(x,\lambda)-\chi_b(x,u)\|_b^2 = \int_{-\infty}^{\infty}\left\{\frac{1}{\pi}\int_0^u \frac{v\,dw}{(w-t)^2+v^2}-s(u,t)\right\}^2 dk_b(t), \quad (5.2)$$

where $s(u,\lambda_{nb})$ is $+1$ if $0<\lambda_{nb}\leqslant u$, -1 if $u<\lambda_{nb}<0$, and vanishes otherwise, so that, as $v\to 0$, the integrand tends to zero if $t\neq u$. To avoid the difficulty of the discontinuous integrand, we assume that u is

a point of continuity of all $k_b(t)$, and of $k(t)$. If $A > 2|u|$, so that $t \geqslant A$ implies $t-w \geqslant \frac{1}{2}t$, then

$$\int_A^\infty \left\{ \int_0^u \frac{v\,dw}{(w-t)^2+v^2} \right\}^2 dk_b(t) \leqslant 4v^2u^2 \int_A^\infty \frac{dk_b(t)}{t^4} \leqslant \frac{v^2Ku^2}{A^2},$$

by (4.2), and similarly for the integral over $(-\infty, -A)$. As

$$\sigma_b(x,\lambda) \to \sigma(x,\lambda), \quad \chi_b(x,u) \to \chi(x,u)$$

for given x, λ, it follows from (5.2) that

$$\|\sigma(x,\lambda)-\chi(x,u)\|_R^2 \leqslant \int_{-A}^{A} \left\{ \frac{1}{\pi} \int_0^u \frac{v\,dw}{(w-t)^2+v^2} - s(u,t) \right\}^2 dk(t) + \frac{v^2Ku^2}{A^2}.$$

For this it is sufficient that u be a point of continuity of $k(t)$, and (iv) follows by $R \to \infty$, $v \to 0$.

As an immediate consequence of (i), (ii) of the lemma, we have

$$\int_0^b \chi_b(x,w)\{\psi_b(x,\lambda)-\psi(x,\lambda)\}\,dx + \int_0^b \psi(x,\lambda)\{\chi_b(x,w)-\chi(x,w)\}\,dx = o(1)$$

as $b \to \infty$, $v > 0$. By the Parseval relation,

$$\int_0^b \chi_b(x,w)\psi_b(x,\lambda)\,dx = \int_0^w \frac{dk_b(t)}{\lambda-t}.$$

Combining these results, we obtain

$$\int_0^\infty \psi(x,\lambda)\chi(x,w)\,dx = \int_0^w \frac{dk(t)}{\lambda-t}, \tag{5.3}$$

$$\int_0^\infty \sigma(x,\lambda)\chi(x,w)\,dx = \frac{1}{\pi}\int_0^w dk(t) \int_0^u \frac{v\,dy}{(y-t)^2+v^2}.$$

By (iv) of the lemma, we may make $v \to 0$ in this result, and this proves (2.2), and the following theorem.

THEOREM 2. *With the meanings assigned above to the functions ρ, χ, ϕ, k, the hypotheses and conclusions of Theorem* 1 *hold in both the limit-point and the limit-circle cases.*

This formulation includes Levinson's Theorems 1 and 2 of (**4**), and the completion of Theorem 2, proved in (**5**) in the limit-point case. Another proof for the restricted case $q(x) \in L$ appears in (**6**).

6. Let $G(x,y,\lambda)$ denote the Green's function, defined as

$$\psi(x,\lambda)\phi(y,\lambda) \quad (y \leqslant x), \qquad \phi(x,\lambda)\psi(y,\lambda) \quad (y > x) \quad (v > 0).$$

Then $G(x,y,\lambda)$ is L^2 $(0 < y < \infty)$. Also, for any $f \in L^2$, $v > 0$,

$$\Phi(x,\lambda,f) = \int_0^\infty G(x,y,\lambda)f(y)\,dy \tag{6.1}$$

is of class L^2.

We write $f \in W$ if $f \in L^2$, $L\{f\} = q(x)f(x) - f''(x) \in L^2$, and $f'(x)$ is an integral, and associate with such a function

$$A = f(0)\cos\alpha + f'(0)\sin\alpha, \tag{6.2}$$

$$\rho(\lambda) = \begin{cases} 0 & \text{(limit-point case)}, \\ W_\infty\{\psi(x,\lambda), f(x)\} & \text{(limit-circle case)}, \end{cases} \tag{6.3}$$

where $W_x\{\psi(x,\lambda), f(x)\} = \psi(x,\lambda)f'(x) - \psi'(x,\lambda)f(x)$. Then, with the usual procedure, $G(x,y,\lambda)$ satisfies the differential equation, except at $y = x$; so, substituting $\lambda G = qG - G''$ in (6.1), and integrating by parts, we obtain, for $v > 0$, $f \in W$,

$$\lambda\,\Phi(x,\lambda,f) = f(x) + \Phi(x,\lambda,Lf) - A\,\psi(x,\lambda) + \rho(\lambda)\phi(x,\lambda), \tag{6.4}$$

the last term being absent in the limit-point case, since then ϕ is not L^2.

Transforms $Tf = F$, $f \in L^2$, $F \in \mathscr{L}^2$

	$f(x)$	$F(t)$
[1]	$1\ (0 < x < a)$, $0\ (x > a)$	$\rho(a,t)$
[2]	$\chi(x,a)$	$\begin{cases} a > 0,\ 1\ (0 \leqslant x \leqslant a),\ 0 \text{ (otherwise)}, \\ a < 0,\ 1\ (a < x \leqslant 0),\ 0 \text{ (otherwise)} \end{cases}$
[3]	$\psi(x,\lambda)$ $(v > 0)$	$1/(\lambda - t)$
[4]	$G(x,y,\lambda)$ $(v > 0)$	$\phi(y,t)/(\lambda - t)$
[5]	$\partial G(x,y,\lambda)/\partial y$ $(v > 0)$	$\phi'(y,t)/(\lambda - t)$
[6]	$\Phi(x,\lambda,f)$ $(v > 0, f \in L^2)$	$F(t)/(\lambda - t)$
[7]	$\phi(x,\lambda)$ (limit-circle case)	$\phi_1(t,\lambda) = \int_0^\infty \phi(x,\lambda)\phi(x,t)\,dx$
[8]	$L\{f(x)\}$ $(f \in W)$	$t\,F(t) + A - (\lambda - t)\rho(\lambda)\phi_1(t,\lambda)$ $(v > 0)$.

Another form for [8] is given by the relation

$$(\lambda - t)\rho(\lambda)\phi_1(t,\lambda) = W_\infty\{\phi(x,t), f(x)\} \tag{6.5}$$

in the limit-circle case. The transform [3] is given by (5.3). By the method used to obtain (5.3) we may obtain [4], [5], beginning with

$$\int_0^b G_b(x,y,\lambda)\chi_b(x,w)\,dx = \int_0^w \frac{\phi(y,t)\,dk_b(t)}{\lambda-t},$$

G_b being the Green's function for $(0,b)$. By [4], and the Parseval relation of Theorem 2, we then have the general form of Theorem 2.17 of (7), for any $f \in L^2$,

$$\Phi(x,\lambda,f) = \int_{-\infty}^{\infty} \frac{F(t)\,\phi(x,t)\,dk(t)}{\lambda-t}, \tag{6.6}$$

the integral being absolutely convergent. Then [6] follows, and [8] and (6.5) are obtained by replacing each function in (6.4) by its transform. Also, on substituting in (6.4) from (6.6), the expansion theorem follows.

THEOREM 3. *Let $f \in W$. Then, for any $\lambda = u+iv$ with $v > 0$,*

$$f(x) = \int_{-\infty}^{\infty} \left\{F(t)+\rho(\lambda)\phi_1(t,\lambda)-\frac{A}{\lambda-t}\right\}\phi(x,t)\,dk(t) + \\ +A\,\psi(x,\lambda)-\rho(\lambda)\phi(x,\lambda).$$

The next four results, required in § 7, are derived by applying to the appropriate functions the Parseval formula of Theorem 2. They are valid for $\mathrm{im}(\lambda) > 0$, $\mathrm{im}(\mu) > 0$, but $\mu = \bar{\lambda}$ is permissible, the functions of μ becoming the conjugates of the functions of λ. Also f, $g \in L^2$, and F, G are their transforms.

$$\int_0^{\infty} \psi(x,\lambda)\psi(x,\mu)\,dx = \int_{-\infty}^{\infty} \frac{dk(t)}{(\lambda-t)(\mu-t)} = \frac{m(\lambda)-m(\mu)}{\mu-\lambda}, \tag{6.7}$$

from (4.1) and the lemma.

$$\int_0^{\infty} G(x,y,\lambda)\,G(\xi,y,\mu)\,dy = \int_{-\infty}^{\infty} \frac{\phi(x,t)\phi(\xi,t)}{(\lambda-t)(\mu-t)}\,dk(t) \\ = \frac{G(x,\xi,\lambda)-G(x,\xi,\mu)}{\mu-\lambda}. \tag{6.8}$$

$$\int_0^{\infty} g(x)\Phi(x,\lambda,f)\,dx = \int_{-\infty}^{\infty} \frac{F(t)G(t)}{\lambda-t}\,dk(t). \tag{6.9}$$

$$\int_0^{\infty} \Phi(x,\lambda,f)\Phi(x,\mu,g)\,dx = \int_{-\infty}^{\infty} \frac{F(t)G(t)}{(\lambda-t)(\mu-t)}\,dk(t). \tag{6.10}$$

7. The spectrum, being the complement of the set of points in the neighbourhood of which $k(t)$ is constant, is a closed set. (Naturally the spectrum in the limit-circle case depends on the sequence through which $b \to \infty$, but the considerations of this section still apply.) Suppose that $\lambda = u+iv$, where u is not in the spectrum. Then integrals with respect to $k(t)$ over $(-\infty, \infty)$ are actually integrals over $(-\infty, u-\delta)$, $(u+\delta, \infty)$, for some $\delta > 0$. Hence, by (4.3), (6.7), $m(u) = \lim_{v \to 0} m(\lambda)$ exists, and

$$\frac{m(u)-m(\mu)}{\mu-u} = \int_{-\infty}^{\infty} \frac{dk(t)}{(u-t)(\mu-t)}, \qquad \mathrm{im}(\mu) > 0.$$

Now define

$$\psi(x,u) = \lim_{v \to 0} \psi(x,\lambda) = \theta(x,u)+m(u)\phi(x,u).$$

By (6.7) with $\mu = \bar{\lambda}$, $\psi(x,u) \in L^2$ since now $\mathrm{im}\{m(\lambda)\} = O(v)$ as $v \to 0$. By the Parseval relation and (4.3), with v, $v' > 0$,

$$\|\psi(x,u+iv)-\psi(x,u+iv')\|^2 = \int_{-\infty}^{\infty} \left|\frac{1}{u+iv-t} - \frac{1}{u+iv'-t}\right|^2 dk(t)$$

$$= O\{|v-v'|^2\}.$$

Making $v' \to 0$, it follows from Fatou's theorem that $\psi(x,\lambda)$ converges to $\psi(x,u)$ in L^2 norm as $v \to 0$. By (5.3), the transform of $\psi(x,u)$ is $1/(u-t)$. A similar argument applies to (6.8), and finally we may define

$$\Phi(x,u,f) = \int_0^{\infty} G(x,y,u)f(y)\,dy = \lim_{v \to 0} \Phi(x,\lambda,f).$$

By (6.10) with $f = g$, $\mu = \bar{\lambda}$, this function is L^2.

THEOREM 4. *If $\lambda = u+iv$, and u is not in the spectrum of* (3.1), *then the results of* § 6 *hold with λ replaced by u; $\psi(x,u) \in L^2$ and satisfies* (3.1) *with $\lambda = u$, and $\Phi(x,u,f)$ is L^2 if $f \in L^2$, and satisfies*

$$\frac{d^2y}{dx^2}+\{u-q(x)\}y = f(x)$$

at all points of continuity of $f(x)$.

The existence of the functions $\psi(x,u)$, $\Phi(x,u,f)$ has been used by Hartman and Wintner (3) to prove, in the limit-point case, the following theorem.

THEOREM 5. *The spectrum of* (3.1) *contains values of t clustering at* $t = \infty$.

With the reservation at the beginning of this section, the theorem holds also for the limit-circle case. What is still required for the proof is (6.9).

For, suppose that the spectrum is empty for $t \geqslant c$. Then, by Sturm's theory, we may fix $u > c$ so large that $\psi(x, u)$ has a zero in $x > 0$, and so an interval (a, b) exists in which $\psi(x, u)$ has one sign and $\phi(x, u)$ the opposite sign. Let f be any real function of L^2, vanishing outside (a, b). We then have, from (6.9) with $g = f$, $\lambda = u$,

$$\int_{-\infty}^{c} \frac{\{F(t)\}^2\, dk(t)}{u-t} = 2 \int_a^b f(x)\psi(x, u)\, dx \int_a^x f(y)\phi(y, u)\, dy.$$

This is impossible, since the integrals differ in sign if $f > 0$.

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SOME PROPERTIES OF EIGENFUNCTION EXPANSIONS

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1. In a previous paper† I considered the summability of the eigenfunction expansion

$$f(x) \sim \sum_{n=0}^{\infty} c_n \psi_n(x), \tag{1.1}$$

where $c_n = \int_0^{\infty} f(t)\psi_n(t)\,dt$, associated with a differential equation

$$\frac{d^2\psi}{dx^2} + \{\lambda - q(x)\}\psi = 0 \quad (0 \leqslant x < \infty), \tag{1.2}$$

with the boundary condition

$$\psi(0)\cos\alpha + \psi'(0)\sin\alpha = 0. \tag{1.3}$$

It is assumed that the spectrum is discrete. The expansion is said to be 'summable (R, λ, p) to $f(x)$' if

$$\lim_{\Lambda\to\infty} \sum_{\lambda_n<\Lambda} \left(1 - \frac{\lambda_n}{\Lambda}\right)^p c_n \psi_n(x) = f(x).$$

It was shown that the expansion is summable $(R, \lambda, 1)$ to $f(x)$ wherever

$$\int_0^{\eta} |f(x+t) - f(x)|\,dt = o(\eta), \tag{1.4}$$

as $\eta \to 0$, and so almost everywhere, provided that $f(x)$ is $L^2(0, \infty)$. In the general case it seems difficult to avoid this condition; but, if $q(x)$ is subject to special conditions, the same result can be obtained for more general functions $f(x)$. If $q(x)$ tends steadily to infinity, each eigenfunction $\psi_n(x)$ is exponentially small at infinity,‡ so that the 'Fourier coefficients' c_n exist in much more general circumstances. Actually I shall show that, if $q(x)$ satisfies certain conditions, the series is

† *Quart. J. of Math.* (Oxford) (2) 2 (1951), 258–68.

‡ See Lemma 10.3 of my book *Eigenfunction Expansions associated with Second-order Differential Equations.*

summable (R, λ, p) almost everywhere if $f(x)$ is integrable over any finite interval, and if, as $T \to \infty$,

$$\int_0^T |f(t)|\, dt = o(T^a) \tag{1.5}$$

with a suitable value of a. The value of a required increases indefinitely with p, so that any condition of this form is sufficient for summability of some order.

With regard to the function $q(x)$, I assume various conditions which make it roughly similar to a polynomial of degree k. No doubt these conditions could be varied considerably without affecting the results.

In the latter part of the paper I consider the extension to these expansions of the Riesz–Fischer and Young–Hausdorff theorems in the theory of Fourier series.

2. In this section I shall assume that $q(x)$ is non-decreasing, and that there are positive numbers A and k such that, as $x \to \infty$,

$$q(x) \sim Ax^k, \qquad q'(x) \asymp x^{k-1}, \qquad q''(x) = O(x^{k-2}), \qquad q'''(x) = O(x^{k-3}). \tag{2.1}$$

Let $\phi(x, \lambda)$ and $\psi(x, \lambda)$ denote, as before, the solutions of (1.2) which satisfy (1.3), and are L^2 up to ∞, respectively. I write as before

$$\Phi(x,\lambda) = \psi(x,\lambda)\left(\int_0^{x-\delta} + \int_{x-\delta}^{x}\right)\phi(t,\lambda)f(t)\,dt + \phi(x,\lambda)\left(\int_x^{x+\delta} + \int_{x+\delta}^{\infty}\right)\psi(t,\lambda)f(t)\,dt$$
$$= \Phi_1 + \Phi_2 + \Phi_3 + \Phi_4.$$

Now, in dealing with Φ_1, Φ_2, and Φ_3, all that we really assume is that $f(t)$ is $L(0, x+\delta)$. The condition that f is $L^2(0, \infty)$ is relevant only to Φ_4, since here the relations (4.2) of the previous paper are used. What we do now is to consider the function $\psi(t, \lambda)$ directly under the special conditions assumed, and we have to consider its behaviour when both t (real) and λ (complex) may be large.

Let $z(t) = \lambda - q(t)$, where $\operatorname{im}\lambda > 0$, and make the substitution

$$\xi(x) = \int_0^x \{z(t)\}^{\frac{1}{2}}\, dt, \qquad \eta(x) = \{z(x)\}^{\frac{1}{4}}\psi(x).$$

Here $\{z(t)\}^{\frac{1}{2}}$ denotes the value of the square root whose imaginary part is large and positive when t is large; and similarly for $\{z(x)\}^{\frac{1}{4}}$. It is then easily seen that $\operatorname{im}\{z(t)\}^{\frac{1}{2}} > 0$ for all t, and hence that $\operatorname{im}\xi(x)$ is a steadily increasing function of x.

The equation (1.2) transforms into

$$\frac{d^2\eta}{d\xi^2}+\eta+\left\{\frac{5z'^2(u)}{16z^3(u)}-\frac{z''(u)}{4z^2(u)}\right\}\eta=0. \tag{2.2}$$

This is formally equivalent to the integral equation

$$\eta(x)=e^{i\xi(x)}+\int_x^\infty \sin\{\xi(t)-\xi(x)\}R(t)\eta(t)\,dt, \tag{2.3}$$

where

$$R(t)=\frac{z''(t)}{4z^{\frac{3}{2}}(t)}-\frac{5z'^2(t)}{16z^{\frac{5}{2}}(t)}.$$

Let $\lambda=\rho e^{i\gamma}$ $(0<\gamma_0\leqslant\gamma\leqslant\pi)$. Then

$$\begin{aligned}|\lambda-q(t)|^2&=\rho^2+q^2(t)-2\rho q(t)\cos\gamma\\&\geqslant(1-\cos\gamma_0)\{\rho^2+q^2(t)\}.\end{aligned}$$

Hence, if γ_0 is fixed, as $\rho\to\infty$,

$$\int_0^\infty |R(t)|\,dt=O\left\{\int_0^\infty\frac{t^{k-2}\,dt}{(\rho^2+t^{2k})^{\frac{3}{4}}}\right\}+O\left\{\int_0^\infty\frac{t^{2k-2}\,dt}{(\rho^2+t^{2k})^{\frac{5}{4}}}\right\}=O(\rho^{-\frac{1}{2}-1/k}). \tag{2.4}$$

We can now solve (2.3) by iteration, putting†

$$\eta_1(x)=e^{i\xi(x)}$$

and

$$\eta_{n+1}(x)=e^{i\xi(x)}+\int_x^\infty \sin\{\xi(t)-\xi(x)\}R(t)\eta_n(t)\,dt\quad(n=1,2,\ldots). \tag{2.5}$$

We obtain

$$|\eta_2(x)-\eta_1(x)|=\left|\int_x^\infty \sin\{\xi(t)-\xi(x)\}R(t)e^{i\xi(t)}\,dt\right|\leqslant e^{-\mathrm{im}\xi(x)}j(x),$$

where

$$j(x)=\int_x^\infty |R(t)|\,dt;$$

and, if

$$|\eta_n(x)-\eta_{n-1}(x)|\leqslant\frac{e^{-\mathrm{im}\xi(x)}j^{n-1}(x)}{(n-1)!},$$

then

$$|\eta_{n+1}(x)-\eta_n(x)|\leqslant\frac{e^{-\mathrm{im}\xi(x)}}{(n-1)!}\int_x^\infty |R(t)|j^{n-1}(t)\,dt=\frac{e^{-\mathrm{im}\xi(x)}j^n(x)}{n!}.$$

Hence $\eta_n(x)$ tends to a limit $\eta(x)$; and

$$|\eta_n(x)|\leqslant e^{-\mathrm{im}\xi(x)+j(x)}\leqslant e^{-\mathrm{im}\xi(x)+j(0)},\qquad|\eta(x)|\leqslant e^{-\mathrm{im}\xi(x)+j(0)}. \tag{2.6}$$

† Cf. D. B. Sears, *Proc. London Math. Soc.* (2) 53 (1951), 396–421, Lemma 1.

It follows by dominated convergence that we can make $n \to \infty$ in (2.5), and so obtain (2.3). From (2.3),

$$\frac{d}{dx}\left|\frac{\eta'(x)}{\xi'(x)}\right| = \{R(x)-\xi'(x)\}\eta(x),$$

and this reduces to (2.2).

Since there is only one solution of (1.2) of $L^2(0,\infty)$,

$$\psi(x,\lambda) = C\{z(x)\}^{\frac{1}{4}}\eta(x), \tag{2.7}$$

where C is independent of x; and from

$$\psi(0,\lambda)\cos\alpha+\psi'(0,\lambda)\sin\alpha = 1$$

we obtain

$$1/C = \{z(0)\}^{-\frac{1}{4}}\eta(0)\cos\alpha+[\{z(0)\}^{-\frac{1}{4}}\eta'(0)-\tfrac{1}{4}z'(0)\{z(0)\}^{-\frac{5}{4}}\eta(0)]\sin\alpha. \tag{2.8}$$

From the above formulae we now obtain, as $\lambda \to \infty$ in the above angle, uniformly with respect to x,

$$\eta(x) \sim e^{i\xi(x)}, \qquad \eta'(x) \sim i\{z(x)\}^{\frac{1}{2}}e^{i\xi(x)}, \qquad 1/C \sim i\lambda^{\frac{1}{4}}\sin\alpha, \tag{2.9}$$

$$\psi(x,\lambda) \sim \frac{e^{i\xi(x)}}{i\lambda^{\frac{1}{4}}\{z(x)\}^{\frac{1}{4}}\sin\alpha}. \tag{2.10}$$

Also $$\operatorname{im} z^{\frac{1}{2}}(t) \geqslant \operatorname{im}\sqrt{\lambda} = \tau,$$

say. Hence $$\operatorname{im}\xi(x) \geqslant x\tau.$$

Since $|z(x)| > A|\lambda|$, it follows that

$$\psi(x,\lambda) = O(|\lambda|^{-\frac{1}{2}}e^{-x\tau}) \tag{2.11}$$

for all x and large λ in the above angle.

If $t > A|\lambda|^{1/k}$, then

$$\operatorname{im} z^{\frac{1}{2}}(t) > At^{\frac{1}{2}k}, \qquad \operatorname{im}\xi(x) > Ax^{\frac{1}{2}k+1},$$

and so $$\psi(x,\lambda) = O\{|\lambda|^{-\frac{1}{2}}\exp(-Ax^{\frac{1}{2}k+1})\} \quad (x > A|\lambda|^{1/k}). \tag{2.12}$$

3. To obtain similar results when $0 \leqslant \arg\lambda < \gamma_0$ I shall use a method due to Langer.† We now suppose that $q(x)$ is an analytic function regular in an angle including the positive real axis and that the conditions (2.1) hold in this angle. The equation $q(x) = \lambda$ then has a solution $x = X = X(\lambda)$ which tends to a real positive value as λ tends to a real positive value. Let

$$\zeta(x) = \int_X^x \{z(t)\}^{\frac{1}{2}}\,dt.$$

† *Trans. American Math. Soc.* 33 (1931), 23–64.

The equation (2.2) is then equivalent to

$$\frac{d^2\eta}{d\zeta^2}+\left(1+\frac{5}{36\zeta^2}\right)\eta = f(\zeta)\eta, \tag{3.1}$$

where $$f(\zeta) = \frac{5}{36\zeta^2}+\frac{z''(x)}{4z^2(x)}-\frac{5z'^2(x)}{16z^3(x)}.$$

A solution of (3.1) is obtained by solving the integral equation

$$\eta = (\tfrac{1}{2}\pi\zeta)^{\frac{1}{2}}H_{\frac{1}{3}}^{(1)}(\zeta)+\tfrac{1}{4}\pi i\int_{\zeta}^{\infty}\{H_{\frac{1}{3}}^{(1)}(\zeta)H_{\frac{1}{3}}^{(2)}(\theta)-H_{\frac{1}{3}}^{(2)}(\zeta)H_{\frac{1}{3}}^{(1)}(\theta)\}\zeta^{\frac{1}{2}}\theta^{\frac{1}{2}}f(\theta)\eta(\theta)\,d\theta,$$

in the usual notation of Bessel functions; i.e.

$$\eta = (\tfrac{1}{2}\pi\zeta)^{\frac{1}{2}}H_{\frac{1}{3}}^{(1)}(\zeta)+\tfrac{1}{4}\pi i\int_{x}^{\infty}\{H_{\frac{1}{3}}^{(1)}(\zeta)H_{\frac{1}{3}}^{(2)}(\theta)-H_{\frac{1}{3}}^{(2)}(\zeta)H_{\frac{1}{3}}^{(1)}(\theta)\}\zeta^{\frac{1}{2}}\theta^{\frac{1}{2}}F(t)\eta(t)\,dt, \tag{3.2}$$

where $\theta = \zeta(t)$ and $F(t) = \{z(t)\}^{\frac{1}{2}}f(\theta)$.

Now

$$\begin{aligned}\zeta(x) &= \int_{X}^{x}\frac{z'(t)\{z(t)\}^{\frac{1}{2}}}{z'(t)}\,dt\\ &= \frac{2\{z(x)\}^{\frac{3}{2}}}{3z'(x)}+\frac{4\{z(x)\}^{\frac{5}{2}}z''(x)}{15z'^3(x)}-\frac{4}{15}\int_{X}^{x}\{z(t)\}^{\frac{5}{2}}\frac{d}{dt}\left(\frac{z''(t)}{z'^3(t)}\right)dt.\end{aligned}$$

If $\frac{1}{2}|X| \leqslant x \leqslant 2|X|$, $\quad z(t) = O(|t-X||X|^{k-1})$,

and the last term on the right-hand side is

$$O\left(\int_{X}^{x}|t-X|^{\frac{5}{2}}|X|^{\frac{1}{2}k-\frac{5}{2}}\,dt\right) = O(|x-X|^{\frac{7}{2}}|X|^{\frac{1}{2}k-\frac{5}{2}}).$$

Since also $|z(t)| > A|t-X||X^{k-1}|$, we obtain

$$\zeta(x) = \frac{2\{z(x)\}^{\frac{3}{2}}}{3z'(x)}\left\{1+\frac{2z(x)z''(x)}{5z'^2(x)}+O\left(\frac{|x-X|^2}{|X|^2}\right)\right\},$$

$$\frac{1}{\zeta^2(x)} = \frac{9z'^2(x)}{4z^3(x)}\left\{1-\frac{4z(x)z''(x)}{5z'^2(x)}+O\left(\frac{|x-X|^2}{|X|^2}\right)\right\},$$

and hence $$F(x) = O(|X|^{-\frac{1}{2}k-\frac{3}{2}}|x-X|^{-\frac{1}{2}}).$$

Hence $$\int_{\frac{1}{2}|X|}^{2|X|}|F(x)|\,dx = O(|X|^{-\frac{1}{2}k-1}).$$

For large x, $F(x) = O(x^{-\frac{1}{2}k-2})$, and so the integral over $(2|X|,\infty)$ is of the same order; and similarly for $(0,\frac{1}{2}|X|)$. Altogether

$$\int_{0}^{\infty}|F(x)|\,dx = O(|X|^{-\frac{1}{2}k-1}) = O(|\lambda|^{-\frac{1}{2}-\frac{1}{2}k}). \tag{3.3}$$

We can now solve (3.2) by iteration. If $\operatorname{im}\lambda > 0$, and x varies from 0 to ∞, $\arg\zeta(x)$ varies from $-\pi$ to $\frac{1}{2}\pi$. Hence we can write

$$e^{-i\zeta}(\tfrac{1}{2}\pi\zeta)^{\frac{1}{2}}H_{\frac{1}{3}}^{(1)}(\zeta) = \alpha(x), \qquad e^{i\zeta}(\tfrac{1}{2}\pi\zeta)^{\frac{1}{2}}H_{\frac{1}{3}}^{(2)}(\zeta) = \beta(x),$$

where $\alpha(x)$ and $\beta(x)$ are bounded in $0 \leqslant x < \infty$. The second of these results follows from § 7.2 (8) of Watson's *Theory of Bessel Functions*. The first follows from § 7.2 (7) except as regards an angle

$$-\pi \leqslant \arg\zeta \leqslant -\pi+\delta.$$

However, if $\zeta = \zeta' e^{-i\pi}$, then

$$H_{\frac{1}{3}}^{(1)}(\zeta) = \frac{2e^{\frac{1}{3}i\pi}}{i\sqrt{3}}\{J_{\frac{1}{3}}(\zeta')+J_{-\frac{1}{3}}(\zeta')\}, \tag{3.4}$$

and the result stated follows from § 7.21 (1).

Writing $e^{-i\zeta}\eta(x) = \chi(x)$, we obtain from (3.2)

$$\chi(x) = \alpha(x)+\tfrac{1}{2}i\int_x^\infty \{\alpha(x)\beta(t)-e^{2i(\theta-\zeta)}\beta(x)\alpha(t)\}F(t)\chi(t)\,dt,$$

where
$$\operatorname{im}\theta-\operatorname{im}\zeta = \operatorname{im}\left\{\int_x^t \{\lambda-q(u)\}^{\frac{1}{2}}\,du\right\} \geqslant 0.$$

Hence the method of iteration leads to a solution $\chi(x)$ such that

$$\chi(x) = \alpha(x)+O(|\lambda|^{-\frac{1}{2}-1/k})$$

uniformly in x; and so

$$\eta(x) = (\tfrac{1}{2}\pi\zeta)^{\frac{1}{2}}H_{\frac{1}{3}}^{(1)}(\zeta)+O(|\lambda|^{-\frac{1}{2}-1/k}e^{-\operatorname{im}\zeta}). \tag{3.5}$$

Similarly, by differentiating the above formulae, we obtain

$$\eta'(x) = z^{\frac{1}{2}}(x)(\tfrac{1}{2}\pi\zeta)^{\frac{1}{2}}H_{\frac{1}{3}}^{(1)\prime}(\zeta)+O(|\lambda|^{-\frac{1}{4}-1/k}e^{-\operatorname{im}\zeta}). \tag{3.6}$$

If $\operatorname{im}\zeta(0) > A$, it follows from the asymptotic formulae for the Bessel functions that

$$C = O(|\lambda|^{-\frac{1}{4}}e^{\operatorname{im}\zeta(0)}),$$

and hence that

$$\psi(x,\lambda) = O(|\lambda|^{-\frac{1}{2}}e^{\operatorname{im}\zeta(0)-\operatorname{im}\zeta(x)}) = O(|\lambda|^{-\frac{1}{2}}e^{-\operatorname{im}\xi(x)}).$$

Hence (2.11) and (2.12) follow again in this case.

If $0 \leqslant \operatorname{im}\zeta(0) < A$, by (3.4)

$$\eta'(0) \sim z^{\frac{1}{2}}(0)(-2i)e^{\frac{1}{3}i\pi}\sin\{\omega(\lambda)-\tfrac{1}{4}\pi\},$$

where
$$\omega(\lambda) = \int_0^X \{\lambda-q(t)\}^{\frac{1}{2}}\,dt.$$

The same results as before follow if $\operatorname{re}\omega(\lambda) = (n-\frac{1}{4})\pi$. Now consider

the function $\phi(\omega) = (\omega-\omega_n)C(\omega)$, where ω_n is the pole of $C(\omega)$ in the square

$$(n-\tfrac{1}{4})\pi \leqslant \operatorname{re}\omega \leqslant (n+\tfrac{3}{4})\pi, \qquad -\tfrac{1}{2}\pi \leqslant \operatorname{im}\omega \leqslant \tfrac{1}{2}\pi.$$

On the perimeter, $\phi(\omega) = O(|\lambda|^{-\frac{1}{4}})$, and so this holds throughout the square. Since ω_n is real, it follows that, for all values of ω,

$$C(\omega) = O\{|\lambda|^{-\frac{1}{4}}|\operatorname{im}\omega|^{-1}\}.$$

We shall therefore obtain the same results as before, but with an extra factor $|\operatorname{im}\omega|^{-1}$.

If $\operatorname{im}\lambda > 0$, $\operatorname{im}X > 0$, we write

$$\omega(\lambda) = \int_0^{\operatorname{re}X} + \int_{\operatorname{re}X}^{X} = \omega_1+\omega_2,$$

each integral being taken along a straight line. *Now suppose that* $\operatorname{im}q(t)$ *increases steadily as* t *varies from* $\operatorname{re}X$ *to* X. Then, in this range

$$0 < \arg\{\lambda-q(t)\} < \pi, \qquad 0 < \arg\{\lambda-q(t)\}^{\frac{1}{2}} < \tfrac{1}{2}\pi,$$

and so $\operatorname{im}\omega_2 > 0$. Since $\operatorname{im}\{\lambda-q(t)\}^{\frac{1}{2}} > 0$ in ω_1,

$$\operatorname{im}\omega_1 > \int_0^{A\operatorname{re}X} \{\lambda-q(t)\}^{\frac{1}{2}}\,dt > A\operatorname{re}X\operatorname{im}\lambda^{\frac{1}{2}}.$$

Now $\operatorname{re}X > A|\lambda|^{1/k}$, and, if $\lambda = \mu+i\nu$,

$$\operatorname{im}\lambda^{\frac{1}{2}} > A\mu^{-\frac{1}{2}}\nu > A|\lambda|^{-\frac{1}{2}}\nu.$$

Hence

$$\operatorname{im}\omega > A|\lambda|^{-\frac{1}{2}+1/k}\nu.$$

It follows that, *if* $0 < \nu < |\lambda|^{\frac{1}{2}-1/k}$, *an additional factor* $|\lambda|^{\frac{1}{2}-1/k}\nu^{-1}$ *must be inserted on the right-hand side of* (2.11) *and* (2.12).

4. Summability theorem

Let $q(x)$ *satisfy all the conditions imposed in the previous sections, and let* $f(x)$ *be integrable over any finite interval and satisfy* (1.5). *Then the series* (1.1) *is summable* (R,λ,p) *to* $f(x)$ *wherever* (1.4) *holds, and so almost everywhere, provided that* $p \geqslant 1$ *and*

$$p \geqslant 2(a-1)/(k+2). \tag{4.1}$$

It is a question of proving that

$$\int_{|\lambda|=\Lambda} \Phi_4(x,\lambda)\left(1-\frac{\lambda}{\Lambda}\right)^p d\lambda$$

tends to zero as $\Lambda \to \infty$. Putting $\lambda = \kappa^2$, $\Lambda = K^2$, it is sufficient to prove that

$$\left(\int_K^{K+iK} + \int_{K+iK}^{iK}\right)\Phi_4(x,\kappa^2)\left(1-\frac{\kappa^2}{K^2}\right)^p \kappa\, d\kappa \to 0. \tag{4.2}$$

The first part is

$$\int_0^K \Phi_4(x,\kappa^2)\left\{-\frac{i\tau}{K}\left(2+\frac{i\tau}{K}\right)\right\}^p (K+i\tau)i\, d\tau$$

$$= O\left\{K^{1-p}\int_0^K |\Phi_4(x,\kappa^2)|\tau^p\, d\tau\right\}$$

$$= O\left\{K^{1-p}\left(\int_0^{1/M} + \int_{1/M}^K\right)\right\} = O\{K^{1-p}(I_1+I_2)\},$$

say, where $M = K^{2k}$. Since $\nu = 2K\tau$, this division corresponds to the two ranges in which an extra factor $(M\tau)^{-1}$ is or is not required.

In I_2,

$$\Phi_4(x,\lambda) = O\left[\frac{e^{x\tau}}{K}\left\{\int_{x+\delta}^{AM} e^{-t\tau}|f(t)|\, dt + \int_{AM}^{\infty} \exp(-At^{\frac{1}{2}k+1})|f(t)|\, dt\right\}\right]. \tag{4.3}$$

The first part contributes to I_2

$$O\left\{\frac{1}{K}\int_0^K e^{x\tau}\tau^p\, d\tau \int_{x+\delta}^{AM} e^{-t\tau}|f(t)|\, dt\right\}$$

$$= O\left\{\frac{1}{K}\int_{x+\delta}^{AM}|f(t)|\, dt \int_0^{\infty} e^{-(t-x)\tau}\tau^p\, d\tau\right\}$$

$$= O\left\{\frac{1}{K}\int_{x+\delta}^{AM}\frac{|f(t)|}{(t-x)^{p+1}}\, dt\right\} = O\left\{\frac{1}{K}\int_{x+\delta}^{AM}\frac{|f(t)|}{t^{p+1}}\, dt\right\},$$

x and δ being fixed. Using (1.5) and integrating by parts, we obtain

$$o(K^{-1}M^{a-p-1}), \qquad o(K^{-1}\log M), \qquad O(K^{-1})$$

according as $a > p+1$, $a = p+1$, or $a < p+1$. The contribution of the second part of (4.3) will clearly be exponentially small. Altogether I_2 contributes to the original integral

$$o(K^{-p}M^{a-p-1}), \qquad o(K^{-p}\log M), \qquad O(K^{-p})$$

in the three cases. Each of these is $o(1)$ if (4.1) holds.

The corresponding term in I_1 is

$$O\left\{\frac{1}{KM}\int_{x+\delta}^{AM}|f(t)|\,dt\int_{0}^{1/M}e^{-(t-x)\tau}\tau^{p-1}\,d\tau\right\}$$

$$=O\left\{\frac{M^{-p}}{KM}\int_{x+\delta}^{AM}|f(t)|\,dt\right\}=o(K^{-1}M^{a-p-1}),$$

and this is $o(1)$ under the same condition.

The remainder of (4.2) contains a factor $e^{-\delta K}$, and so is negligible. This completes the proof.

5. Extension of the Parseval and Riesz–Fischer theorems

Consider any problem of the above type with a discrete spectrum, and let $f(x)$ be $L^2(0,\infty)$. Then the usual 'Bessel inequality' argument shows that

$$\sum_{n=0}^{\infty}c_n^2\leqslant\int_0^{\infty}\{f(x)\}^2\,dx.$$

Actually the case of equality always holds (Parseval theorem); and in fact, if

$$s_n(x)=\sum_{m=0}^{n}c_m\psi_m(x),$$

then $s_n(x)$ converges in mean square to $f(x)$ over $(0,\infty)$.

To obtain a theorem of the converse (Riesz–Fischer) type, suppose that $q(x)$ satisfies the conditions under which the above summability theorem holds, and that $f(x)$ is integrable over any finite interval, and satisfies (1.5), with some given a. Then $\sum c_n\psi_n(x)$ is summable (R,λ,p) to $f(x)$ almost everywhere, with some p.

Now suppose that $\sum c_n^2$ is convergent. Then $s_n(x)$ converges in mean square over $(0,\infty)$, say to $g(x)$; $g(x)$ is $L^2(0,\infty)$, and the numbers c_n are its Fourier coefficients, since

$$\int_0^{\infty}g(x)\psi_m(x)\,dx=\lim_{n\to\infty}\int_0^{\infty}s_n(x)\psi_m(x)\,dx$$

and

$$\int_0^{\infty}s_n(x)\psi_m(x)\,dx=c_m\quad(n\geqslant m).$$

Hence $\sum c_n\psi_n(x)$ is summable $(R,\lambda,1)$ to $g(x)$ almost everywhere, by the theorem of the previous paper. Hence $g(x)=f(x)$ almost everywhere.

Thus, *if $f(x)$ is integrable over any finite interval and satisfies* (1.5) *with some a, and $\sum c_n^2$ is convergent, then $f(x)$ is $L^2(0,\infty)$.*

6. Extension of the Young–Hausdorff theorems

The object of the remaining sections is to extend to a general class of eigenfunction expansions the Young–Hausdorff theorems in the theory of Fourier series. We base the theory on the following general theorem of M. Riesz† on the convexity of linear operations.

Consider a system of numbers

$$a_{jl} \quad (1 \leqslant j \leqslant m,\ 1 \leqslant l \leqslant n)$$

and the linear forms

$$X_j = a_{j1}x_1 + \ldots + a_{jn}x_n \quad (j = 1, \ldots, m)$$

of the variables $x_1, \ldots, x_n$. Let

$$M_{\alpha,\beta} = \max_{x_1,\ldots,x_n} \Big(\sum_{j=1}^{m} \sigma_j |X_j|^{1/\beta} \Big)^{\beta} \Big(\sum_{l=1}^{n} \rho_l |x_l|^{1/\alpha} \Big)^{-\alpha},$$

where σ_j and ρ_l are arbitrary but fixed positive numbers. This maximum is attained for every positive α and β.

The theorem of M. Riesz is that $M_{\alpha,\beta}$ *is a multiplicatively convex function of the variables* α, β *in the triangle* $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant \alpha$. This means that, if (α, β), (α_1, β_1), (α_2, β_2) are collinear points in the triangle such that

$$\alpha = \alpha_1 t_1 + \alpha_2 t_2, \qquad \beta = \beta_1 t_1 + \beta_2 t_2 \qquad (t_1 + t_2 = 1;\ t_1, t_2 > 0),$$

then

$$M_{\alpha,\beta} \leqslant M_{\alpha_1,\beta_1}^{t_1} M_{\alpha_2,\beta_2}^{t_2}. \tag{6.1}$$

A proof is given by Zygmund [*Trigonometrical Series*, 192–4].

Now suppose that the function $q(x)$ satisfies the conditions imposed in the previous sections. Then the eigenfunctions $\psi_n(x)$ satisfy

$$|\psi_n(x)| < A\lambda_n^h \quad \left(h = \frac{1}{12} - \frac{1}{3k}\right) \tag{6.2}$$

for all n and x. This result is due to W. E. Milne.‡ It follows from equations (23) and (26) of his paper, together with formula (7.2.2) of my book. More refined formulae showing that h is the best possible constant were obtained by D. O. H. Weekes in an Oxford D.Phil. thesis.

The functions $$\phi_n(x) = \lambda_n^{-h}\psi_n(x)$$

are therefore bounded for all n and x.

Let $$c_j = \int_0^\infty \psi_j(t) f(t)\, dt, \qquad X_j = \int_0^\infty \phi_j(t) f(t)\, dt$$

for any $f(t)$ such that the integrals exist. Then $X_j = \lambda_j^{-h} c_j$.

† *Acta Math.* 49 (1926), 465–97.

‡ *Trans. American Math. Soc.* 31 (1929), 907–18.

Consider first step-functions $f(t)$ with steps at given points $t_1,\ldots,\ t_n$, $0 < t_1 < \ldots < t_n$, and vanishing for $t > t_n$. Let

$$f_l(t) = \begin{cases} 1 & (t_{l-1} < t \leqslant t_l), \\ 0 & \text{elsewhere.} \end{cases}$$

Then
$$f(t) = x_1 f_1(t) + \ldots + x_n f_n(t)$$

with some set of x's. Also

$$|f(t)|^{1/\alpha} = |x_1|^{1/\alpha} f_1(t) + \ldots + |x_n|^{1/\alpha} f_n(t),$$

$$\int_0^\infty |f(t)|^{1/\alpha}\, dt = |x_1|^{1/\alpha}(t_1 - t_0) + \ldots + |x_n|^{1/\alpha}(t_n - t_{n-1}).$$

Let $\rho_l = t_l - t_{l-1}$ in the definition of $M_{\alpha,\beta}$, so that the denominator in $M_{\alpha,\beta}$ is

$$\left\{ \int_0^\infty |f(t)|^{1/\alpha}\, dt \right\}^\alpha.$$

Let
$$\beta = 1-\alpha, \qquad \alpha_1 = 1, \qquad \beta_1 = 0, \qquad \alpha_2 = \tfrac{1}{2}, \qquad \beta_2 = \tfrac{1}{2},$$
$$t_1 = 2\alpha - 1, \qquad t_2 = 2(1-\alpha).$$

Then
$$M_{\alpha,1-\alpha} \leqslant M_{1,0}^{2\alpha-1} M_{\frac{1}{2},\frac{1}{2}}^{2(1-\alpha)}.$$

Now
$$M_{1,0} = \max_{x_l} \left\{ \max |X_j| \Big/ \int_0^\infty |f(t)|\, dt \right\} < A$$

since the functions $\phi_j(t)$ are bounded. Also

$$M_{\frac{1}{2},\frac{1}{2}} = \max_{x_l} \left(\sum_{j=1}^m \sigma_j X_j^2 \right)^{\frac{1}{2}} \Big/ \left\{ \int_0^\infty f^2(t)\, dt \right\}^{\frac{1}{2}}.$$

Let $\sigma_j = \lambda_j^{2h}$. Then $\sigma_j X_j^2 = c_j^2$, and it follows from the Bessel inequality that $M_{\frac{1}{2},\frac{1}{2}} \leqslant 1$. Hence $M_{\alpha,1-\alpha} < A$, i.e.

$$\left(\sum_{j=1}^m \lambda_j^{h(2-1/\beta)} |c_j|^{1/\beta} \right)^\beta < A \left(\int_0^\infty |f(t)|^{1/\alpha}\, dt \right)^\alpha. \tag{6.3}$$

This inequality holds for all step-functions vanishing outside a finite interval, and all values of m.

Finally, let $f(t)$ be any function of the class $L^{1/\alpha}(0,\infty)$. Then we can define a sequence of functions $f_\nu(t)$ of the above type such that, as $\nu \to \infty$,

$$\int_0^\infty |f_\nu(t)|^{1/\alpha}\, dt \to \int_0^\infty |f(t)|^{1/\alpha}\, dt,$$

while $c_{j,\nu} \to c_j$ for each j. We therefore obtain (6.3) for any such function. Since m is arbitrary, we obtain the following theorem:

THEOREM. *If $f(t)$ is any function of the class $L^{1/\alpha}(0,\infty)$, where $\frac{1}{2} \leqslant \alpha < 1$, then $\sum \lambda_n^{h(2-1/\beta)} |c_n|^{1/\beta}$, where $\beta = 1-\alpha$, is convergent; and*

$$\Big(\sum_{n=0}^{\infty} \lambda_n^{h(2-1/\beta)} |c_n|^{1/\beta}\Big)^{\beta} < A\Big(\int_0^{\infty} |f(t)|^{1/\alpha}\, dt\Big)^{\alpha}. \tag{6.4}$$

7. To prove a theorem of the opposite type, let $f(x)$ first denote a finite sum of the form

$$f(x) = \sum c_n \psi_n(x).$$

Such a sum belongs to all L-classes over $(0,\infty)$. Let

$$g(x) = |f(x)|^{\alpha/\beta} \operatorname{sgn} f(x)$$

and let d_n be the 'Fourier coefficient' of $g(x)$. Then

$$\begin{aligned}\int_0^{\infty} |f(x)|^{1/\beta}\, dx = \int_0^{\infty} f(x) g(x)\, dx &= \sum_{n=0}^{\infty} c_n d_n \\ &= \sum_{n=0}^{\infty} \lambda_n^{h(1-2\beta)} c_n \lambda_n^{h(2\beta-1)} d_n \\ &\leqslant \Big\{\sum_{n=0}^{\infty} |\lambda_n^{h(1-2\beta)} c_n|^{1/\alpha}\Big\}^{\alpha} \Big\{\sum_{n=0}^{\infty} |\lambda_n^{h(2\beta-1)} d_n|^{1/\beta}\Big\}^{\beta}.\end{aligned}$$

By the above theorem, the last factor does not exceed

$$A\Big(\int_0^{\infty} |g(x)|^{1/\alpha}\, dx\Big)^{\alpha} = A\Big(\int_0^{\infty} |f(x)|^{1/\beta}\, dx\Big)^{\alpha}.$$

Hence

$$\Big(\int_0^{\infty} |f(x)|^{1/\beta}\, dx\Big)^{\beta} < A\Big(\sum_{n=0}^{\infty} \lambda_n^{h(2-1/\alpha)} |c_n|^{1/\alpha}\Big)^{\alpha}. \tag{7.1}$$

THEOREM. *Let $f(x)$ be any function which is integrable over any finite interval, and satisfies* (1.5) *with some a. Let it have 'Fourier coefficients' c_n such that $\sum \lambda_n^{h(2-1/\alpha)} |c_n|^{1/\alpha}$ is convergent. Let $q(x)$ satisfy conditions such that the theorem of* § 4 *holds. Then $f(x)$ belongs to $L^{1/\beta}(0,\infty)$, and* (7.1) *holds.*

By the theorem of § 4,

$$f(x,\Lambda) = \sum_{\lambda_n < \Lambda} \Big(1 - \frac{\lambda_n}{\Lambda}\Big)^p c_n \psi_n(x) \to f(x)$$

almost everywhere if p is large enough. By the above result,

$$\begin{aligned}\Big(\int_0^{\infty} |f(x,\Lambda)|^{1/\beta}\, dx\Big)^{\beta} &< A\Big\{\sum_{\lambda_n < \Lambda} \lambda_n^{h(2-1/\alpha)} \Big(1 - \frac{\lambda_n}{\Lambda}\Big)^{p/\alpha} |c_n|^{1/\alpha}\Big\}^{\alpha} \\ &< A\Big(\sum_{n=0}^{\infty} \lambda_n^{h(2-1/\alpha)} |c_n|^{1/\alpha}\Big)^{\alpha}.\end{aligned}$$

Making $\Lambda \to \infty$ and using Fatou's theorem, we obtain the result stated.

ON THE INTEGRABILITY OF FUNCTIONS DEFINED BY TRIGONOMETRIC SERIES

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[Received 13 October 1953]

THE aim of this note is to show that two theorems of Boas (**1**) hold under conditions wider than those assumed by Boas.

Suppose that λ_n is positive for all values of n and decreases steadily to zero as n tends to infinity. Then in any interval $0 < \delta \leqslant x \leqslant 2\pi - \delta$ the series

$$\tfrac{1}{2}\lambda_0 + \sum_1^\infty \lambda_n \cos nx, \qquad \sum_1^\infty \lambda_n \sin nx$$

converge uniformly [Zygmund (**7**), 3]. Denote their sums by $f(x)$ and $g(x)$ respectively.

Boas proves that, for $0 < \gamma \leqslant 1$, $x^{-\gamma}g(x)$ is $L(0,\pi)$ if and only if $\sum n^{\gamma-1}\lambda_n$ converges. This theorem had previously been proved for $\gamma = 0$ by Young (**6**); we shall see that it is true also for $1 < \gamma < 2$ (Theorem 1). It is false for larger values of γ. For, if $\sum n^{\gamma-1}\lambda_n$ converges, where γ is greater than 2, the work of Hartman and Wintner (**4**) shows that

$$g(x) \sim x\Big(\sum_1^\infty n\lambda_n\Big)$$

as x tends to zero, and so $x^{-\gamma}g(x)$ is not $L(0,\pi)$.

Boas also shows that, for $0 < \gamma < 1$, the convergence of $\sum n^{\gamma-1}\lambda_n$ is necessary and sufficient for the integrability of $x^{-\gamma}f(x)$.† For $\gamma \geqslant 1$, the function $x^{-\gamma}f(x)$ cannot be $L(0,\pi)$ unless $f(0)$ vanishes, and in that case λ_n cannot be positive for all values of n. In his theorem for $\gamma = 1$, Boas therefore assumes that λ_n is ultimately positive and steadily decreasing, and that the series

$$\tfrac{1}{2}\lambda_0 + \sum_1^\infty \lambda_n \tag{1}$$

is convergent to zero. He then proves that $f(x)/x$ is $L(0,\pi)$ if and only if $\sum \lambda_n \log n$ converges. When we consider larger values of γ, we shall assume that λ_n is non-negative for large values of n, and that the series (1) converges to zero. We shall then see that the value $\gamma = 1$ is exceptional, and that, for values of γ between 1 and 3, the convergence of

† The term $\frac{1}{2}\lambda_0$ in the series which defines $f(x)$ is immaterial when γ is less than 1.

Quart. J. Math. Oxford (2), 5 (1954), 71–76.

$\sum n^{\gamma-1}\lambda_n$ is again necessary and sufficient for the integrability of $x^{-\gamma}f(x)$ (Theorem 2). The convergence of the series which defines $f(x)$ is now ensured by that of the series (1), and we need not assume that λ_n is ultimately monotonic.

To see that Theorem 2 breaks down for $\gamma \geqslant 3$, we apply the theorem of Hartman and Wintner (**4**) to the function

$$\phi(x) = \sum_1^\infty n\lambda_n \sin nx,$$

in the case where $n\lambda_n$ decreases steadily to zero and $\sum n^2\lambda_n$ is convergent. It follows that

$$\phi(x) \sim x\Big(\sum_1^\infty n^2\lambda_n\Big)$$

as x tends to zero. If we now write

$$\lambda_0 = -2\sum_1^\infty \lambda_n,$$

an integration which can easily be justified shows that

$$f(x) \sim -\tfrac{1}{2}x^2\Big(\sum_1^\infty n^2\lambda_n\Big).$$

Hence $x^{-\gamma}f(x)$ is not $L(0,\pi)$.

We shall need the following lemma in the proof of Theorem 1.

LEMMA A. *Suppose that μ_{2n} and μ_{2n+1} decrease steadily to zero as n increases from 1 to infinity, that γ lies between 1 and 2, and that $\sum n^{\gamma-2}\mu_n$ converges. Then, if*

$$\lambda_n = \mu_{n-1} - \mu_{n+1} \quad (n = 2, 3, \ldots),$$

the series $\sum n^{\gamma-1}\lambda_n$ is convergent.

Proof. If $\sum a_n$ is a convergent series of positive, decreasing terms, na_n tends to zero as n tends to infinity [Bromwich (**2**), 31]. Applying this result to each of the series

$$\sum_1^\infty (2n)^{\gamma-2}\mu_{2n-1}, \qquad \sum_1^\infty (2n)^{\gamma-2}\mu_{2n},$$

in which the terms decrease steadily as a consequence of our hypotheses, we see that $n(2n)^{\gamma-2}\mu_{2n-1}$ and $n(2n)^{\gamma-2}\mu_{2n}$ tend to zero. It follows that $n^{\gamma-1}\mu_n$ tends to zero. We therefore have, by partial summation,

$$\sum_2^\infty n^{\gamma-1}\lambda_n = 2^{\gamma-1}\mu_1 + 3^{\gamma-1}\mu_2 + \sum_3^\infty \mu_n\{(n+1)^{\gamma-1} - (n-1)^{\gamma-1}\}.$$

But $$(n+1)^{\gamma-1}-(n-1)^{\gamma-1} = 2(\gamma-1)n^{\gamma-2}+o(n^{\gamma-2}),$$

and $\sum n^{\gamma-2}\mu_n$ is given to be convergent. Hence $\sum n^{\gamma-1}\lambda_n$ converges, and the lemma is established.

THEOREM 1. *Suppose that λ_n is positive for all values of n, and decreases steadily to zero as n tends to infinity. Let*

$$g(x) = \sum_1^\infty \lambda_n \sin nx.$$

Then, for $1 < \gamma < 2$, $x^{-\gamma}g(x)$ is $L(0,\pi)$ if and only if

$$\sum_1^\infty n^{\gamma-1}\lambda_n$$

converges.

Proof. First suppose that $\sum n^{\gamma-1}\lambda_n$ converges. Then we can show that $x^{-\gamma}g(x)$ is $L(0,\pi)$ by a method similar† to the one which Boas uses in proving his Theorem 1, but the following proof is more direct.

Since $\sum n^{\gamma-1}\lambda_n$ is convergent, where γ is greater than 1, it follows that $\sum \lambda_n$ converges. Therefore

$$|g(x)| \leqslant \sum_1^\infty \lambda_n |\sin nx|.$$

Hence $$I \equiv \int_0^\pi |g(x)|x^{-\gamma}\,dx \leqslant \int_0^\pi \Big\{ \sum_1^\infty \lambda_n |\sin nx| \Big\} x^{-\gamma}\,dx,$$

and, since the terms of the series are non-negative, the last expression is equal‡ to

$$\sum_1^\infty \lambda_n \int_0^\pi x^{-\gamma}|\sin nx|\,dx.$$

If we write $y = nx$, the last integral becomes

$$n^{\gamma-1}\int_0^{n\pi} y^{-\gamma}|\sin y|\,dy.$$

Hence $$I \leqslant \sum_1^\infty n^{\gamma-1}\lambda_n \int_0^\infty y^{-\gamma}|\sin y|\,dy. \tag{2}$$

The integral on the right converges at the lower limit since γ is less than 2, and converges at the upper limit since γ is greater than 1. The series $\sum n^{\gamma-1}\lambda_n$ converges by hypothesis. Hence the right-hand side of (2) is finite, and so $x^{-\gamma}g(x)$ is $L(0,\pi)$.

† The only change is in the proof of Boas's lemma, the result of which is still true.

‡ At this stage it is not asserted that either expression is finite.

Conversely, suppose that $x^{-\gamma}g(x)$ is $L(0,\pi)$. Let

$$G(x) = g(x)/2\sin x.$$

Then it is clear that $x^{1-\gamma}G(x)$ is L over any interval interior to $(0,\pi)$. But, as x tends to zero,

$$G(x) \sim g(x)/2x,$$

and so $x^{1-\gamma}G(x)$ is L in a neighbourhood of 0. As x tends to π,

$$G(x) \sim g(x)/2(\pi-x). \tag{3}$$

Now γ is greater than 1, and we have assumed that $x^{-\gamma}g(x)$ is $L(0,\pi)$. Hence $g(x)/x$ is $L(0,\pi)$, and so, by Boas's Theorem 1, $\sum \lambda_n$ is convergent. Therefore, by a remark of Boas [(**1**), 219], $g(x)/(\pi-x)$ is L in a neighbourhood of π. It follows from (3) that $G(x)$ is L in a neighbourhood of π. Therefore $x^{1-\gamma}G(x)$ is L in such a neighbourhood. But we know from above that $x^{1-\gamma}G(x)$ is L in a neighbourhood of 0 and over any interval interior to $(0,\pi)$. Hence $x^{1-\gamma}G(x)$ is $L(0,\pi)$.

Since γ is greater than 1, it now follows that $G(x)$ is $L(0,\pi)$. Hence, if

$$\mu_n = \frac{2}{\pi}\int_0^\pi G(x)\cos nx\,dx \quad (n = 0, 1, 2,\ldots),$$

μ_n tends to zero as n tends to infinity, by the Riemann–Lebesgue theorem. Moreover, since $g(x)$ is $L(0,\pi)$, $\sum \lambda_n \sin nx$ is the Fourier series of its sum [Hardy and Rogosinski (**3**), Theorem 46]. Therefore,

$$\begin{aligned}\lambda_n &= \frac{2}{\pi}\int_0^\pi g(x)\sin nx\,dx \\ &= \frac{2}{\pi}\int_0^\pi G(x)\{\cos(n-1)x-\cos(n+1)x\}\,dx \\ &= \mu_{n-1}-\mu_{n+1}.\end{aligned}$$

Hence, since λ_n is positive, μ_{n-1} is greater than μ_{n+1} for all values of n. But μ_n tends to zero. Therefore μ_{2n} and μ_{2n+1} tend to zero monotonically from above, and so μ_n is positive for all values of n.

Dirichlet's test [Bromwich (**2**), 59] now shows that the series

$$\sum_1^\infty \mu_{2n}\cos 2nx, \qquad \sum_1^\infty \mu_{2n-1}\cos(2n-1)x$$

converge for $0 < x < \pi$. Therefore the series

$$\tfrac{1}{2}\mu_0 + \sum_1^\infty \mu_n \cos nx \tag{4}$$

converges for $0 < x < \pi$, to a function $G^*(x)$ say. But the series (4) is summable $(C, 1)$ to $G(x)$ almost everywhere in $(0, \pi)$ by the Féjer–Lebesgue theorem [Titchmarsh (**5**), (13.34)]. Hence $G(x) = G^*(x)$ for almost all x in $(0, \pi)$. Therefore, by our previous work,

$$\mu_n = \frac{2}{\pi} \int_0^\pi G^*(x) \cos nx \, dx$$

and $x^{1-\gamma} G^*(x)$ is $L(0, \pi)$.

But μ_n is positive, and $\gamma - 1$ lies between 0 and 1. Hence, by Boas's Theorem 2, in which the necessity of the condition is proved with the hypothesis that the coefficients are non-negative rather than monotonic. $\sum n^{\gamma-2}\mu_n$ converges. Therefore, by Lemma A, $\sum n^{\gamma-1}\lambda_n$ converges.

This completes the proof of Theorem 1.

THEOREM 2. *Suppose that λ_n is ultimately non-negative, that the series*

$$\tfrac{1}{2}\lambda_0 + \sum_1^\infty \lambda_n$$

converges to zero, and that

$$f(x) = \tfrac{1}{2}\lambda_0 + \sum_1^\infty \lambda_n \cos nx.$$

Then, for $1 < \gamma < 3$, $x^{-\gamma} f(x)$ is $L(0, \pi)$ if and only if

$$\sum_1^\infty n^{\gamma-1} \lambda_n$$

converges.

Proof. Since $\sum \lambda_n$ converges and since λ_n is ultimately non-negative, $\sum \lambda_n$ is absolutely convergent. Therefore $\sum \lambda_n(1 - \cos nx)$ is absolutely convergent, and

$$f(x) = \tfrac{1}{2}\lambda_0 + \sum_1^\infty \lambda_n - \sum_1^\infty \lambda_n(1 - \cos nx)$$

$$= -\sum_1^\infty \lambda_n(1 - \cos nx).$$

First suppose that $\sum n^{\gamma-1}\lambda_n$ converges. Then

$$J \equiv \int_0^\pi x^{-\gamma} |f(x)| \, dx = \int_0^\pi x^{-\gamma} \Big| \sum_1^\infty \lambda_n(1 - \cos nx) \Big| \, dx$$

$$\leqslant \int_0^\pi \sum_1^\infty |\lambda_n| (1 - \cos nx) x^{-\gamma} \, dx,$$

and, since the terms of the last series are non-negative, this is equal to

$$\sum_1^\infty |\lambda_n| \int_0^\pi x^{-\gamma}(1 - \cos nx) \, dx.$$

If we write $y = nx$, the last integral becomes

$$n^{\gamma-1} \int_0^{n\pi} y^{-\gamma}(1-\cos y)\, dy.$$

Hence
$$J \leqslant \sum_1^\infty n^{\gamma-1}|\lambda_n| \int_0^\infty y^{-\gamma}(1-\cos y)\, dy.$$

The integral on the right converges at the lower limit since γ is less than 3, and converges at the upper limit since γ is greater than 1. Also, λ_n is ultimately non-negative, so that the convergence of $\sum n^{\gamma-1}\lambda_n$ implies that of $\sum n^{\gamma-1}|\lambda_n|$. Hence $x^{-\gamma}f(x)$ is $L(0, \pi)$.

Conversely, suppose that $x^{-\gamma}f(x)$ is $L(0, \pi)$. We have

$$J' \equiv -\int_0^\pi x^{-\gamma}f(x)\, dx = \int_0^\pi \sum_1^\infty \lambda_n(1-\cos nx)x^{-\gamma}\, dx.$$

Now, except perhaps for a finite number of terms, each of which is finitely integrable, the terms of the series on the right are non-negative. We may therefore integrate term by term, and we find that

$$J' = \sum_1^\infty \lambda_n \int_0^\pi x^{-\gamma}(1-\cos nx)\, dx$$

$$= \sum_1^\infty n^{\gamma-1}\lambda_n \int_0^{n\pi} y^{-\gamma}(1-\cos y)\, dy.$$

Now J' is finite by hypothesis, and so the last series must be convergent. But, since $y^{-\gamma}(1-\cos y)$ is $L(0, \infty)$, and since λ_n is ultimately non-negative, the terms of this convergent series are ultimately not less than those of the series

$$\tfrac{1}{2}\sum_1^\infty n^{\gamma-1}\lambda_n \int_0^\infty y^{-\gamma}(1-\cos y)\, dy.$$

Hence $\sum n^{\gamma-1}\lambda_n$ converges, and the proof of Theorem 2 is complete.

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SPACES WITH WEAK TOPOLOGY

By D. E. COHEN (*Oxford*)

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1. Introduction

In this note I consider spaces which have the weak topology with respect to a given family of topologized subsets. This means that a set is closed if and only if it meets each of these subsets in a closed set.

In § 2 I consider general properties; § 3 concerns 'K-spaces' (spaces with the weak topology by their compact sets) and in it I find a condition for the product of two K-spaces to be a K-space. I find also a condition for a closure-finite cell complex on a K-space to be a CW-complex and a condition for two CW-complexes on the same set of points to define the same topology.

I wish to thank Professor J. H. C. Whitehead for suggesting the subject of this note and for valuable help.

2. General properties

Let Σ be a family of topological spaces, which I shall denote by S, S_1, S_2, etc., and let X be the point-set-union of all the spaces of Σ. The spaces of Σ are not necessarily disjoint, and we assume that:

[2.1] *S_1, S_2 induce the same topology on $S_1 \cap S_2$ for every pair S_1, $S_2 \in \Sigma$.*

Then we define the weak topology on X, with respect to Σ, as follows. A subset A of X is closed (open) for the weak topology if and only if $A \cap S$ is closed (open) in S for every $S \in \Sigma$. This clearly defines a topology which I shall denote by t_Σ.

The condition [2.1] is satisfied by the members S' of any subfamily $\Sigma' \subset \Sigma$. Therefore $t_{\Sigma'}$ is defined on the union X' of the sets of Σ'. In particular

[2.2] *If every $S \in \Sigma$ is contained in some $S' \in \Sigma'$, then $X' = X$ and $t_{\Sigma'} = t_\Sigma$.*

The given topology of any set S is obviously weaker† than the one induced by t_Σ, and we have

[2.3] *If $S_1 \cap S_2$ is closed (open) in S_1, for every pair S_1, $S_2 \in \Sigma$, then t_Σ induces the given topology on every $S \in \Sigma$.*

† If t_1, t_2 are two topologies on a space X we say that t_1 is *weaker* than t_2 if every set which is closed for t_2 is also closed for t_1.

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[2.4] *If there exists one topology, t say, on X, which induces the given topology on every $S \in \Sigma$, then t_Σ does so, and t_Σ is weaker than t.*

I leave the verification of [2.3] to the reader and proceed to prove [2.4]. Let F be closed for t. Then, for every $S \in \Sigma$, $F \cap S$ is closed for the given topology of S. Therefore F is closed for t_Σ, and t_Σ is therefore weaker than t.

Let $S \in \Sigma$ and let $C \subset S$ be closed for the given topology of S. Then there is a set F such that $F \cap S = C$, and F is closed for t, and therefore closed for t_Σ. Conversely $F \cap S$ is closed for the given topology of S if F is closed for t_Σ. Therefore t_Σ induces on every $S \in \Sigma$ its given topology.

Define a space P as follows. P consists of all pairs (S, x) where $S \in \Sigma$, $x \in S$. For a fixed S, the map $(S, x) \to x$ is to be a homeomorphism onto S (with the given topology), and the set $\{(S, x) \mid x \in S\}$ for fixed S is to be both open and closed in P. This clearly defines a topology. Let $f\colon P \to X$ be the map $(S, x) \to x$.

[2.5] *t_Σ is the identification topology defined by $f\colon P \to X$.*

Clearly f is onto. If $A \subset X$, then $F^{-1}A = \bigcup (S, x)$ for all $S \in \Sigma$, $x \in A \cap S$. Then $f^{-1}A$ is open in P if and only if, for every $S \in \Sigma$,

$$\{(S, x) \mid x \in A \cap S\}$$

is open in $\{(S, x) \mid x \in S\}$, i.e. if and only if $A \cap S$ is open for the given topology of S for every $S \in \Sigma$. This holds if and only if A is open for t_Σ. This proves [2.5]. I shall call f the *natural identification*.

Suppose that X, X' have the weak topologies defined by families of subsets Σ, Σ'. Let $f\colon P \to X$, $f'\colon P' \to X'$ be the natural identifications. We may define two topologies on $X \times X'$, viz. $t_\Sigma \times t_{\Sigma'}$, the product topology, and $t_{\Sigma \times \Sigma'}$, the weak topology by the family $\Sigma \times \Sigma'$ of sets $S \times S'$ with $S \in \Sigma$, $S' \in \Sigma'$. This latter is clearly the identification topology by $g\colon P \times P' \to X \times X'$ where $g(p, p') = (fp, f'p')$.

[2.6] *$t_{\Sigma \times \Sigma'}$ is weaker than $t_\Sigma \times t_{\Sigma'}$. If every point of X is in the interior with respect to t_Σ of a set of Σ, and X' satisfies the similar condition, then $t_{\Sigma \times \Sigma'} = t_\Sigma \times t_{\Sigma'}$.*

Since the family of sets $U \times U'$, with U, U' open for t_Σ, $t_{\Sigma'}$ respectively, forms a base for the open sets of $t_\Sigma \times t_{\Sigma'}$, if $U \times U'$ is open in $t_{\Sigma \times \Sigma'}$, then $t_{\Sigma \times \Sigma'}$ is weaker than $t_\Sigma \times t_{\Sigma'}$. But

$$g^{-1}(U \times U') = (f^{-1}U) \times (f'^{-1}U')$$

is the product of open sets of P and P', and so is open in $P \times P'$. Therefore $U \times U'$ is open in $t_{\Sigma \times \Sigma'}$, which is therefore weaker than $t_\Sigma \times t_{\Sigma'}$.

Let $(x, x') \in B$, where B is open for $t_{\Sigma\times\Sigma'}$, and let $x \in U \subset S$, $x' \in U' \subset S'$, $S \in \Sigma$, $S' \in \Sigma'$, where U is open for t_Σ, U' for $t_{\Sigma'}$.

Then $(x, x') \in B \cap (U \times U') \subset S \times S'$, and $B \cap (U \times U')$ is open in $S \times S'$. Therefore $(x, x') \in V \times V' \subset B \cap (U \times U')$, where V, V' are open for the given topologies of S, S' respectively. Let S_1 be any set of Σ. Then $V \cap S_1$ is open in $S \cap S_1$, and, since $V \subset U \subset S$, $V \cap S_1$ is open in $U \cap S_1$. But U is open for t_Σ, and therefore $U \cap S_1$ is open in S_1. Therefore $V \cap S_1$ is open in S_1, and, since S_1 is an arbitrary set of Σ, V is open for t_Σ. Similarly V' is open for $t_{\Sigma'}$. Therefore $V \times V'$ is open for $t_\Sigma \times t_{\Sigma'}$, and, since $(x, x') \in V \times V' \subset B$, it follows that B is open for $t_\Sigma \times t_{\Sigma'}$ and $t_{\Sigma\times\Sigma'} = t_\Sigma \times t_{\Sigma'}$.

3. K-spaces

A K-space is a space which has the weak topology defined by the family of its compact sets. Clearly compact spaces, locally compact spaces, and CW-complexes are K-spaces.

[3.1] *Let X_0 be a space, X the same set of points with the weak topology by the compact sets of X_0. Then the topology of X is weaker than that of X_0, the spaces X and X_0 have the same compact sets, and X is a K-space.*

By [2.4], X has a weaker topology than X_0. Therefore the identity map $X \to X_0$ is continuous and a set compact in X is compact in X_0. Since the compact sets of X_0 retain their topology in X by [2.4], X and X_0 have the same compact sets. Therefore X is a K-space.

[3.2] *Let X, Y be K-spaces whose compact sets are regular, one of which (say Y) is locally compact. Then $X \times Y$ is a K-space.*

Let f: $P \to X$, g: $Q \to Y$ be the natural identifications, inducing f': $P \times Y \to X \times Y$, g': $P \times Q \to P \times Y$, and h: $P \times Q \to X \times Y$.

Because every compact set of $X \times Y$ is contained in the product of compact sets of X and Y, it follows from [2.2] that $X \times Y$ is a K-space if it has the weak topology with respect to the family of sets $C \times F$, where C, F are compact sets of X, Y respectively. But this holds if $X \times Y$ has the identification topology of h: $P \times Q \to X \times Y$.

Every set in Y is saturated with respect to the identity $Y \to Y$. Therefore it follows from (**2**) [Lemma 4, 1131] that $X \times Y$ has the identification topology of f': $P \times Y \to X \times Y$ if, for any $y \in Y$ and $U \subset Y$ with U open and $y \in U$, there is an open set W with $\overline{W}$ compact and $y \in W \subset \overline{W} \subset U$. This holds since Y is locally compact and (clearly) regular. Similarly, since P is locally compact and regular (clearly), $P \times Y$ has the identification topology of g': $P \times Q \to P \times Y$. Therefore $h^{-1}A = g'^{-1}(f'^{-1}A)$ is

closed in $P \times Q$ if and only if $f'^{-1}A$ is closed in $P \times Y$, which holds if and only if A is closed in $X \times Y$. Therefore $X \times Y$ has the identification topology of h.

The result [3.2] generalizes the result that the product of two CW-complexes, one of which is locally finite (a condition equivalent to local compactness) is a CW-complex [(**3**), result **H**, 227]. An example due to Dowker [(**1**), 563] of two CW-complexes, neither of which is locally finite, whose product is not a CW-complex also (by [3.3] below) shows that the product of two regular K-spaces, neither of which is locally compact, may not be a K-space.

[3.3] *Let K be a closure-finite cell complex which is a Hausdorff K-space. Then K is a CW-complex if and only if every compact set of K is contained in a finite subcomplex.*

[3.4] *Let K and L be two CW-complexes on the same set of points. Then K and L have the same topology if and only if any characteristic map for any cell of one is continuous in the topology of the other.*

I leave the proofs to the reader, remarking only that [3.4] follows easily from (**3**) [result A, 224], (**2**) [Lemma 3, 1131], and the fact that K and L have the same topology if and only if the identity maps $K \to L$ and $L \to K$ are continuous.

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COMPOSITE MATRICES

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1. Introduction

It has been proved by Williamson (**1**) that, if a matrix $\mathbf{F}$, of order mn, is partitioned into an array of m^2 submatrices $\mathbf{f}_{ij}$, of order n, each of which is a rational function $\mathbf{f}_{ij} = f_{ij}(\mathbf{a})$, of non-singular denominator, of some matrix $\mathbf{a}$, with characteristic values $\alpha_1, \ldots, \alpha_n$, then the mn characteristic values of the matrix $\mathbf{F}$ are the mn characteristic values of the n mth order matrices $\mathbf{f}^{(k)}$, where the matrix $\mathbf{f}^{(k)}$ has (i,j)th element $f_{ij}(\alpha_k)$. For the case $n = 1$, the theorem merely asserts that the characteristic values of $\mathbf{F}$ are the characteristic values of $\mathbf{F}$, and thus it has no content; for the case $m = 1$ it becomes the theorem of Sylvester (**2**) and Frobenius (**3**) on the characteristic values of a rational function of a matrix, with an additional statement in regard to multiplicities. My investigation will show a proposition of the same form as this in respect of the conclusion, but where the hypothesis consists just in the commutativity of the partitioned components, which in the case here is ensured by their all being rational functions of some matrix. To state this more explicitly, Williamson's result will appear as a consequence of the theorem that, if $\mathbf{A}$ is a matrix of order mn partitioned into m^2 submatrices $\mathbf{a}_{ij}$ of order n which all commute, then the mn characteristic values of $\mathbf{A}$ are the characteristic values of n mth order matrices $\boldsymbol{\alpha}^{(k)}$, where the matrix $\boldsymbol{\alpha}^{(k)}$ has (i,j)th element $\alpha_{ij}^{(k)}$, and $\alpha_{ij}^{(k)}$ is the kth characteristic value of $\mathbf{a}_{ij}$ in some ordering.

Elsewhere (**4**) there has been occasion to use the identity

$$\left| -\begin{pmatrix} \mathbf{0} & \mathbf{1} & \mathbf{x} \\ \mathbf{1} & \mathbf{a} & \mathbf{u} \\ \mathbf{y} & \mathbf{v} & \mathbf{b} \end{pmatrix} \right| = |-(-\mathbf{b}+\mathbf{vx}+\mathbf{yu}-\mathbf{yax})|,$$

where the partitioned matrices $\mathbf{a}$ and $\mathbf{0}$ are square and of the same order. The negative signs are required to avoid explicit reference to the matrix orders, and to display more clearly the form of a twice-repeated determinant operation. When the matrices involved are all of order unity, this identity reduces to the vacuous statement of definition for a determinant. In its general case, however, the form of a determinant operation involving matrices as elements can be properly seen, though to make

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this interpretation it is required that the unit matrices be considered as freed of having any fixed order, so that the multiplications in which they would appear might be regarded as defined. But this determinant can be taken in a normal way when the matrices are all of the same order, some convention having been given for the order of the multiplications of the factors in each term, should these factors not be commutative. The considered identity then shows a twice-repeated determinant operation on the right as equivalent to a simple one on the left, and suggests the form of a more general theorem, which will appear here as one of our results.

These remarks indicate the grounds for the suggestion which led to this investigation. Its object is a study of determinants and matrices whose elements are commutative matrices, and certain sets of matrices which appear as analogues of the characteristic values of a simple matrix, the ground field being algebraically closed. It has as a part of its necessary background certain results concerning commutative matrices, some account of which must be given first.

2. Properties of commutative matrices†

Characteristic values which belong to a common latent vector of any matrices are said to be *latently associated*, and their association to be *through* that vector. A set of values which are latently associated characteristic values relative to some matrix set is called a *characteristic set* of that matrix set. *Every commutative matrix set has at least one characteristic set*; also *any characteristic set of a commutative matrix set can be extended to a give a characteristic set of any commutative extension of that matrix set*. Thus the characteristic sets of a commutative matrix set can be formed by taking extensions of each of the characteristic sets of any subset. In particular, we have that each characteristic value of each matrix in a commutative set is an element in some characteristic set of that matrix set. *The matrices of any commutative matrix set can all be reduced to triangular form by the same unitary transformation*, and the sets of similarly situated diagonal elements in any such set of simultaneously triangular unitary transforms are the characteristic sets

† The material in this section is abstracted from an account of commutative matrices to be found in a thesis by the author (1953, Bodleian Library, Oxford). Since there is not the space here to develop the proofs in full, and it has not been found possible to dispose of them all by references to the literature, parts of the results are proved here, and indication is given of the lines by which others can be reached, while relevant material to be found elsewhere is indicated in the references.

of the matrix set with repetitions according to certain fixed multiplicities. Through these repetitions the *multiplicity* of a characteristic set can be obtained so as to generalize the concept of the multiplicity of a characteristic value of a single matrix as its multiplicity as a root of the characteristic equation. *The characteristic values of any rational function of a set of commutative matrices are the values of that function when the argument ranges over the characteristic sets*, and the multiplicity of any characteristic value is the sum of the multiplicities of the distinct characteristic sets which are thus associated with it. For any characteristic set of a commutative matrix set there is a subspace of the fundamental vector space defined, which is called the *principal space*, relative to the matrix set, for that characteristic set. These principal spaces determine a resolution of the fundamental vector space into a union of independent subspaces, each with rank equal to the multiplicity of the characteristic set to which it corresponds, and containing all the common latent vectors of the matrices which belong to that characteristic set. Now the multiplicity of a characteristic set is the rank of its corresponding principal space; moreover, the *rank* of a characteristic set is defined as the rank of the space formed by the common latent vectors of the matrices which belong to it. Thus we have that *the rank of a characteristic set is at most its multiplicity*, and also that *latent vectors which belong to distinct characteristic sets are independent*. Each matrix of the set is completely reduced on any principal space, and those of its characteristic values which it has as a transformation of that space are all equal to its characteristic value in the characteristic set corresponding to the space. Thus, by transforming the fundamental space to a space where the fundamental vectors all lie in these principal spaces, the matrices of the commutative set can be simultaneously transformed into conformable direct sums of matrices, each with equal characteristic values, and with order equal to the rank of the principal space on which it operates, or equivalently the multiplicity of the corresponding characteristic set. A unitary transformation, which completely reduces on each principal space, can then simultaneously reduce all these matrices to triangular form, such that each has equal diagonal elements. Accordingly, *the matrices of any commutative set can be simultaneously transformed into conformable direct sums of triangular matrices each having its order equal to the multiplicity and its diagonal elements all equal to the associated element of the characteristic set to which it belongs.*

These are the necessary properties, and I shall now show the lines along which they are demonstrated. The columns of a matrix I call its

vectors, and the rows its *dual vectors*. I denote by $\mathfrak{F}$ the fundamental vector space, and call its base vectors the *fundamental* vectors, the matrix with these as its vectors being the unit matrix. A square matrix is coordinate with a linear transformation in the fundamental vector space. Thus, if $\mathbf{a}$ is a matrix and $\mathbf{X}$ a vector, then $\mathbf{X}$ has the image $\mathbf{aX}$ in the corresponding transformation; moreover, if T is any linear transformation in $\mathfrak{F}$, and $\mathbf{a}$ is the matrix whose vectors are the images under T of the fundamental vectors, then $\mathbf{a}$ is the matrix coordinate with T. Let $\mathfrak{X}$ be a subspace of $\mathfrak{F}$ and $\mathbf{a}$ a matrix such that $\mathbf{a}\mathfrak{X} \subset \mathfrak{X}$; for which we equivalently say that $\mathbf{a}$ has $\mathfrak{X}$ as an invariant space, or $\mathbf{a}$ is reduced on $\mathfrak{X}$. Let $\mathbf{P}$ be a base for $\mathfrak{X}$, that is to say a matrix whose vectors form a maximal linearly independent subset of the vectors of $\mathfrak{X}$; so $\mathbf{Pt} = \mathbf{0}$ only if the vector $\mathbf{t} = \mathbf{0}$, and the vector $\mathbf{X}_\mathbf{t} = \mathbf{Pt}$ simply describes $\mathfrak{X}$ as $\mathbf{t}$ describes its own fundamental space. We call $\mathbf{t}$, whose order equals the rank of $\mathfrak{X}$, the *parametric vector* of $\mathfrak{X}$, relative to the base $\mathbf{P}$. Let $\mathbf{a}^*$ be the matrix whose vectors are the parameters relative to $\mathbf{P}$ of the images of the vectors of $\mathbf{P}$ under $\mathbf{a}$; then $\mathbf{aP} = \mathbf{Pa}^*$. We define $\mathbf{a}^*$ as the *value* of $\mathbf{a}$ on its invariant space $\mathfrak{X}$, relative to the base $\mathbf{P}$; and the values of $\mathbf{a}$ relative to the various bases for $\mathfrak{X}$ are equivalent matrices. If $\mathbf{t}^*$ is any latent vector of $\mathbf{a}^*$, and α the corresponding characteristic value, so that $\mathbf{a}^*\mathbf{t}^* = \mathbf{t}^*\alpha$, then, equivalently, $\mathbf{X}^* = \mathbf{Pt}^*$ is a latent vector of $\mathbf{a}$, and α the corresponding characteristic value: for

$$\mathbf{aX}^* = \mathbf{aPt}^* = \mathbf{Pa}^*\mathbf{t}^*,$$

so that $\mathbf{aX}^* = \mathbf{X}^*\alpha$ if and only if $\mathbf{a}^*\mathbf{t}^* = \mathbf{t}^*\alpha$, since the vectors of $\mathbf{P}$ are independent. Thus the characteristic values of $\mathbf{a}$ on $\mathfrak{X}$ are the characteristic values of $\mathbf{a}^*$; *a matrix has a latent vector on any space which it leaves invariant.*

To show the existence of a characteristic set for any commutative matrix set, and also the extension property for characteristic sets, we prove that, if commutative matrices $\mathbf{a}$, $\mathbf{b}$,... in a set have characteristic values $\alpha, \beta, \ldots$ which form a characteristic set and if $\mathbf{f}$ is a further matrix which commutes with each of these matrices, then there exists a characteristic value ϕ of $\mathbf{f}$ such that the values $\alpha, \beta, \ldots, \phi$ form a characteristic set for the matrix set $\mathbf{a}, \mathbf{b}, \ldots, \mathbf{f}$. Thus, let $\mathfrak{X}$ be the space of common latent vectors of $\mathbf{a}$, $\mathbf{b}$,... through which $\alpha, \beta, \ldots$ are latently associated; so that, for any vector $\mathbf{X}$ of $\mathfrak{X}$, we have $\mathbf{aX} = \mathbf{X}\alpha, \mathbf{bX} = \mathbf{X}\beta, \ldots$. Then, since $\mathbf{f}$ commutes with each of $\mathbf{a}$, $\mathbf{b}$,..., we have

$$\mathbf{afX} = \mathbf{faX} = \mathbf{fX}\alpha, \qquad \mathbf{bfX} = \mathbf{fbX} = \mathbf{fX}\beta, \qquad \ldots;$$

which is to say that $\mathbf{fX}$ belongs to $\mathfrak{X}$ if $\mathbf{X}$ does, or that $\mathfrak{X}$ is invariant for $\mathbf{f}$. Accordingly, there exists a vector $\mathbf{Y}$ in $\mathfrak{X}$ which is a latent vector of $\mathbf{f}$; so let ϕ be the corresponding characteristic value. Then $\mathbf{Y}$ is a common latent vector of $\mathbf{a}$, $\mathbf{b}$,..., $\mathbf{f}$, and α, β,..., ϕ are the corresponding characteristic values. It follows that any $m+1$ commutative matrices have a characteristic set which extends any characteristic set which should exist for some m of them. An inductive application of this proposition shows the general extension property of characteristic sets. Since a single matrix has at least one characteristic value, we deduce that any finite commutative matrix set has at least one characteristic set; and, since any infinite matrix set has a finite generating subset, we can conclude the validity of this result for infinite sets.

Now let $\mathbf{f} = f(\mathbf{s})$ denote a rational function of a commutative matrix set $\mathbf{s}$, and let any of its characteristic sets be denoted by σ. If $\mathbf{X}$ is any common latent vector of the members of $\mathbf{s}$ which belongs to σ, we have $f(\mathbf{s})\mathbf{X} = \mathbf{X}f(\sigma)$. Thus $f(\sigma)$ is a characteristic value of $f(\mathbf{s})$ for each characteristic set σ of $\mathbf{s}$. Moreover, let ϕ be any characteristic value of $\mathbf{f}$; then, by the extension property of characteristic sets, there exists a characteristic set σ of $\mathbf{s}$ which is such that (ϕ, σ) is a characteristic set of the commutative matrix set $(\mathbf{f}, \mathbf{s})$; so let $\mathbf{Y}$ be any latent vector which belongs to it. Then $\mathbf{fY} = \mathbf{Y}\phi$, and $f(\mathbf{s})\mathbf{Y} = \mathbf{Y}f(\sigma)$, where $\mathbf{f} = f(\mathbf{s})$; so that $\phi = f(\sigma)$. It follows that all, and hence exactly all, the characteristic values of $f(\mathbf{s})$ are included among the values of $f(\sigma)$ when σ ranges over the characteristic sets of $\mathbf{s}$; which is the required result of Frobenius.

I shall now prove inductively the theorem on the simultaneous transformation of commutative matrices to triangular form. This theorem is verified in the case of order one; thus it remains to show it for any commutative matrices $\mathbf{a}$, $\mathbf{b}$,... of order $m+1$ on the hypothesis of its validity in the case of order m. Let $\mathbf{X}$ be a common unit latent vector of the matrices, which we know exists, and α, β,... the corresponding characteristic values; and let $\mathbf{U}_0$ be a unitary matrix with its initial vector $\mathbf{X}$, and the rest of its unit vectors any orthogonal set orthogonal to $\mathbf{X}$. Then

$$\mathbf{U}_0^{-1}\mathbf{a}\mathbf{U}_0 = \begin{pmatrix} \alpha & \boldsymbol{\xi} \\ \mathbf{0} & \mathbf{a}_1 \end{pmatrix}, \qquad \mathbf{U}_0^{-1}\mathbf{b}\mathbf{U}_0 = \begin{pmatrix} \beta & \boldsymbol{\eta} \\ \mathbf{0} & \mathbf{b}_1 \end{pmatrix}, \qquad \ldots,$$

as can easily be seen. Since $\mathbf{a}$, $\mathbf{b}$,... commute, so do the mth-order matrices $\mathbf{a}_1$, $\mathbf{b}_1$,..., and, by the inductive hypothesis, there exists a unitary matrix $\mathbf{U}_1$ which simultaneously transforms them to triangular

form. Thus we have a unitary matrix

$$\mathbf{U} = \mathbf{U}_0 \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{U}_1 \end{pmatrix}$$

which simultaneously transforms $\mathbf{a}$, $\mathbf{b}$,... to triangular form.

It can be shown that, for any matrix $\mathbf{a}$, with characteristic values α, there exist unique matrices $\mathbf{e}_\alpha$ which commute with $\mathbf{a}$, whose sum is the unit matrix, and are such that $\mathbf{e}_\alpha \mathbf{e}_\lambda = \mathbf{0}$ $(\lambda \neq \alpha)$ and $(\mathbf{a}-\alpha\mathbf{1})\mathbf{e}_\alpha$ is nullpotent. They are idempotents, and are thus coordinate with projections of the fundamental space. They are called the *principal projectors* relative to $\mathbf{a}$, and their ranges are identical with the principal spaces relative to $\mathbf{a}$. Each can be exhibited as a rational function, and hence they are expressible as polynomials, in $\mathbf{a}$. Thus, if

$$\phi(\zeta) = \prod_\alpha (\zeta-\alpha)^{i_\alpha}$$

is the minimum function of $\mathbf{a}$, let

$$\phi_\alpha(\zeta) = \prod_{\lambda \neq \alpha} (\zeta-\lambda)^{i_\lambda} / \sum_\mu \prod_{\lambda \neq \mu} (\zeta-\lambda)^{i_\lambda}.$$

Then it can be shown that $\mathbf{e}_\alpha = \phi_\alpha(\mathbf{a})$.

Let $\mathbf{f}_\alpha, \mathbf{g}_\beta,...$ be the principal projectors relative to commutative matrices $\mathbf{a}, \mathbf{b},...$, for characteristic values $\alpha, \beta,...$ respectively. These all commute since they are expressible as polynomials of matrices which all commute. It can be shown that $\mathbf{f}_\alpha \mathbf{g}_\beta... \neq \mathbf{0}$ if and only if $\alpha, \beta,...$ are latently associated. If $\mathbf{s}$ denotes the matrix set with members $\mathbf{a}, \mathbf{b},...$, and σ its characteristic set formed by latently associated characteristic values $\alpha, \beta,...$, we thus have a set of projectors $\mathbf{e}_\sigma = \mathbf{f}_\alpha \mathbf{g}_\beta...$ which are mutually independent, whose sum is the unit matrix, and which are such that $\{\mathbf{f}(\mathbf{s})-f(\sigma)\mathbf{1}\}\mathbf{e}_\sigma$ is nullpotent for any rational function $f(\mathbf{s})$ of $\mathbf{s}$. The ranges of these projectors give the principal spaces relative to $\mathbf{s}$ and are formed of intersections of the principal spaces for the members of $\mathbf{s}$ which belong to latently associated characteristic values. On each of these spaces each matrix of the set is completely reduced and has just a single distinct characteristic value.

The existence of at least one common latent vector for any set of commutative matrices and hence the possibility of their simultaneous transformation to triangular form was at least suspected by Frobenius; proofs, and some further discussions, are given by Drazin, Dungay, and Gruenberg (**5**), though from a rather different point of view. The theorem on the characteristic values of rational functions of commutative matrices, formulated in terms of some existing association of the

characteristic values, though without exhibiting this as through their belonging to common latent vectors, was proved by Frobenius (**6**), and further proofs were given subsequently by Phillips (**7**), MacDuffee (**8**), and Rademacher (**9**). For relevant discussions concerning projectors and commutative matrices, reference may be made to Wedderburn (**10**), Hamburger and Grimshaw (**11**), and Halmos (**12**). The theorem on the simultaneous reduction of commutative matrices into direct sums of matrices with equal characteristic values was proved in a different way by Plemelj (**13**).

3. Definitions and propositions

A *composite matrix* of *order* n and *degree* m is an nth-order matrix whose elements, called its *components*, are matrices of order m. Let the (i,j)th component $\mathbf{A}^*_{ij}$ of a composite matrix $\mathbf{A}^*$, of order n and degree m, have (r,s)th element A^{rs}_{ij}. Then, in a reciprocal relation to $\mathbf{A}^*$, there is defined a dual composite matrix $\mathbf{A}_*$, of order m and degree n, whose (r,s)th component $\mathbf{A}^{rs}_*$ has (i,j)th element A^{rs}_{ij}. Any composite matrix has a *ground matrix*, which is that simple matrix from which it can be derived by partitioning. The ground matrix of a product of composite matrices is the product of their ground matrices and the dual is the product of their duals. Also, when the inverse is defined, the dual of the inverse of a composite matrix is the inverse of its dual. A composite matrix is called *commutative* if its components all commute, and *doubly commutative* if its dual is also commutative. The determinant of a commutative composite matrix, of order n and degree m, is a determinant, of order n, whose value is a matrix of order m; and the determinant of this determinant defines the *double-determinant* of that composite matrix. When the components of a commutative composite matrix are replaced by their respective characteristic values in a characteristic set of the commutative set of these components, the matrix of order n obtained defines a *characteristic dual component* of the composite matrix, with a multiplicity attached to it equal to the multiplicity of the characteristic set formed by its elements. The characteristic dual components of a commutative composite matrix are linearly dependent on its dual components. When a composite matrix is doubly commutative, a set of *characteristic components* is similarly defined; and these, and the characteristic dual components, form commutative matrix sets, for which the arrays formed by their characteristic sets, repeated, in a suitable order, according to their multiplicities, are transposes of each other, and define the *characteristic arrays* for the dual composite pair.

Among the principal results are the following. The characteristic values of the ground matrix of a commutative composite matrix are the characteristic values of its characteristic dual components, with multiplicities determined correspondingly. Thus the ground characteristic values of a doubly commutative composite matrix are the elements of its characteristic arrays. The double determinant of a commutative composite matrix is equal to the determinant of its ground matrix; by which we have a twice-repeated determinant operation as equivalent to a simple one. Let $\boldsymbol{\alpha}^*$, $\boldsymbol{\alpha}_*$ denote characteristic components of a commutative dual pair $\mathbf{A}^*$, $\mathbf{A}_*$ and $m_{\boldsymbol{\alpha}^*}$, $m_{\boldsymbol{\alpha}_*}$ the multiplicities attached to them; and let $\boldsymbol{\lambda}^*$, $\boldsymbol{\lambda}_*$ denote indeterminates in the algebras generated by the respective components. The composite characteristic polynomials are defined by

$$\mathbf{C}^*(\boldsymbol{\lambda}^*) = |\boldsymbol{\lambda}^*\mathbf{1}^{**} - \mathbf{A}^*|, \qquad \mathbf{C}_*(\boldsymbol{\lambda}_*) = |\boldsymbol{\lambda}_*\mathbf{1}_{**} - \mathbf{A}_*|,$$

where $\mathbf{1}^{**}$, $\mathbf{1}_{**}$ denote dual composite matrices whose components are unit matrices on the diagonal and are null elsewhere. Then

$$\mathbf{C}^*(\boldsymbol{\alpha}^*) = \mathbf{0}^*, \qquad \mathbf{C}_*(\boldsymbol{\alpha}_*) = \mathbf{0}_*,$$

and, further,

$$\mathbf{C}^*(\boldsymbol{\lambda}^*) = \prod_{\boldsymbol{\alpha}^*} (\boldsymbol{\lambda}^* - \boldsymbol{\alpha}^*)^{m_{\boldsymbol{\alpha}^*}}, \qquad \mathbf{C}_*(\boldsymbol{\lambda}_*) = \prod_{\boldsymbol{\alpha}_*} (\boldsymbol{\lambda}_* - \boldsymbol{\alpha}_*)^{m_{\boldsymbol{\alpha}_*}}.$$

The Cayley–Hamilton theorem shows that

$$\mathbf{C}^*(\mathbf{A}^*) = \mathbf{0}^{**}, \qquad \mathbf{C}_*(\mathbf{A}_*) = \mathbf{0}_{**},$$

and it is deduced that

$$\prod_{\boldsymbol{\alpha}^*} (\mathbf{A}^* - \boldsymbol{\alpha}^*\mathbf{1}^{**})^{m_{\boldsymbol{\alpha}^*}} = \mathbf{0}^{**}, \qquad \prod_{\boldsymbol{\alpha}_*} (\mathbf{A}_* - \boldsymbol{\alpha}_*\mathbf{1}_{**})^{m_{\boldsymbol{\alpha}_*}} = \mathbf{0}_{**}.$$

4. Demonstrations

THEOREM I. *The ground matrix of a product of composite matrices is the product of their ground matrices, and the dual is the product of their duals.*

If $\mathbf{A}^*$, $\mathbf{B}^*$ is any pair of composite matrices, of the same order and degree, their product is a similar composite matrix $\mathbf{C}^* = \mathbf{A}^*\mathbf{B}^*$. If $\mathbf{a}^*$, $\mathbf{b}^*$, $\mathbf{c}^*$ denote the ground matrices of $\mathbf{A}^*$, $\mathbf{B}^*$, $\mathbf{C}^*$ respectively, then the theorem [(14) 24] on the multiplication of partitioned matrices shows that $\mathbf{c}^* = \mathbf{a}^*\mathbf{b}^*$. Moreover the (i,j)th component of $\mathbf{C}^*$ is

$$\mathbf{C}^*_{ij} = \sum_k \mathbf{A}^*_{ik}\mathbf{B}^*_{kj},$$

with (r,s)th element $C^{rs}_{ij} = \sum A^{rt}_{ik} B^{ts}_{kj}$ which is the (i,j)th element of the (r,s)th component $\mathbf{C}^{rs}_*$ of the dual $\mathbf{C}_*$ of $\mathbf{C}^*$. Thus $\mathbf{C}^{rs}_* = \sum \mathbf{A}^{rt}_*\mathbf{B}^{ts}_*$, which is the (r,s)th component of the product $\mathbf{A}_*\mathbf{B}_*$, so that $\mathbf{C}_* = \mathbf{A}_*\mathbf{B}_*$.

These results can now be extended by induction to products with any finite number of factors.

THEOREM II. *If the determinant of a commutative composite matrix is non-singular, then it has an inverse such that the ground matrix of the inverse composite matrix is the inverse of its ground matrix, and the dual of the inverse is the inverse of the dual.*

For a simple matrix $\mathbf{a}$ and its adjugate adj $\mathbf{a}$, we have, as an expression of Laplace's theorem, the polynomial identities

$$\mathbf{a}(\text{adj}\,\mathbf{a}) = (\text{adj}\,\mathbf{a})\mathbf{a} = |\mathbf{a}|\mathbf{1}$$

in its elements; and, if the components of a composite matrix $\mathbf{A}^*$ are commutative, they may be substituted for these elements, to give corresponding identities

$$\mathbf{A}^*(\text{adj}\,\mathbf{A}^*) = (\text{adj}\,\mathbf{A}^*)\mathbf{A}^* = |\mathbf{A}^*|\mathbf{1}^{**}.$$

If the determinant $|\mathbf{A}^*|$ of $\mathbf{A}^*$ is non-singular, that is to say its double determinant is non-zero, we may define the composite matrix

$$\mathbf{B}^* = \text{adj}\,\mathbf{A}^*/|\mathbf{A}^*|,$$

thus given by the same formula as the inverse of a simple matrix, which is such that $\mathbf{A}^*\mathbf{B}^* = \mathbf{B}^*\mathbf{A}^* = \mathbf{1}^{**}$, and defines the inverse composite matrix $\mathbf{A}^{*-1}$ of $\mathbf{A}^*$. It is commutative, and its components all commute with those of $\mathbf{A}^*$. If $\mathbf{a}^*$, $\mathbf{b}^*$ are the ground matrices of $\mathbf{A}^*$, $\mathbf{B}^*$, then Theorem I shows that $\mathbf{a}^*\mathbf{b}^* = \mathbf{b}^*\mathbf{a}^* = \mathbf{1}$, or that $\mathbf{a}^*$, $\mathbf{b}^*$ are inverse matrices. We deduce, incidentally, that the determinant of a commutative composite matrix is non-singular if and only if the ground determinant is non-singular, which can also be seen directly from a subsequent theorem. Now, on taking duals, Theorem I shows that

$$\mathbf{A}_*\mathbf{B}_* = \mathbf{B}_*\mathbf{A}_* = \mathbf{1}_{**},$$

i.e. that the inverse of the dual also exists, and is the dual of the inverse.

Let $\mathbf{A}^*$ for now denote a commutative composite matrix of order n and degree m, and let elements denoted by α_{ij}^{μ} ($\mu = 1,\ldots,m$) form the characteristic sets of its commutative set of components $\mathbf{A}_{ij}$, enumerated in some order by μ with repetitions equal to their multiplicities as characteristic sets. Then the characteristic dual components of $\mathbf{A}^*$ are the m matrices $\boldsymbol{\alpha}_*^{\mu}$ ($\mu = 1,\ldots,m$), of order n, with (i,j)th elements α_{ij}^{μ} respectively. A common latent vector $\mathbf{X}_{\mu}^*$ of the components of $\mathbf{A}^*$, corresponding to the characteristic set formed by the elements of $\boldsymbol{\alpha}_*^{\mu}$, that is to say a vector $\mathbf{X}_{\mu}^*$ such that $\mathbf{A}_{ij}^*\,\mathbf{X}_{\mu}^* = \mathbf{X}_{\mu}^*\,\alpha_{ij}^{\mu}$, is called a *component*

latent vector of $\mathbf{A}^*$, corresponding to the characteristic dual component $\boldsymbol{\alpha}^\mu_*$; and its elements are denoted by $\mathbf{X}^*_{\mu,r}$.

THEOREM III. *If $\mathbf{A}^*$ is a commutative composite matrix, and $\mathbf{X}^*_\mu$ is a component latent vector corresponding to the characteristic dual component $\boldsymbol{\alpha}^\mu_*$, then*

$$\sum_s \mathbf{A}^{rs}_* X^*_{\mu,s} = \boldsymbol{\alpha}^\mu_* X^*_{\mu,r}.$$

For we have $$\mathbf{A}^*_{ij}\mathbf{X}^*_\mu = \mathbf{X}^*_\mu \alpha^\mu_{ij},$$

by the definition of $\mathbf{X}^*_\mu$; which is $\sum_s A^{rs}_{ij} X^*_{\mu,s} = X^*_{\mu,r}\alpha^\mu_{ij}$, and this may be written

$$\sum_s \mathbf{A}^{rs}_* X^*_{\mu,s} = X^*_{\mu,r}\boldsymbol{\alpha}^\mu_*.$$

COROLLARY I. *The characteristic dual components of a commutative composite matrix are linear combinations of the dual components.*

This is exhibited directly by the theorem, which shows that any characteristic dual component can be expressed as a linear combination of the rth row of dual components if $\mathbf{X}^*_{\mu,r} \neq 0$; which will be the case for some r since $\mathbf{X}^*_\mu \neq \mathbf{0}$.

COROLLARY II. *The characteristic and the characteristic dual components of a doubly commutative composite matrix form commutative matrix sets.*

This is an immediate consequence of Corollary I.

THEOREM IV. *The characteristic values of the ground matrix of a commutative composite matrix are the characteristic values of its characteristic dual components, and multiplicities are determined correspondingly.*

We have a commutative composite matrix $\mathbf{A}^*$ of order n and degree m, with the ground matrix $\mathbf{a}^*$, and characteristic dual components $\boldsymbol{\alpha}^\mu_*$ ($\mu = 1,\ldots,m$), and it is required to prove that the mn characteristic values of $\mathbf{a}^*$ are those of the m nth-order matrices $\boldsymbol{\alpha}^\mu_*$. Since $\mathbf{A}^*$ is commutative, then, by the first transformation theorem for commutative matrices given in § 2, its components all have triangular transforms by some unitary matrix $\mathbf{U}$ such that α^μ_{ij} appears as the μth diagonal element in the transform of $\mathbf{A}^*_{ij}$. Let $\mathbf{U}^*$ denote the diagonal composite matrix, of order n, whose components are equal to $\mathbf{U}$: that is to say $\mathbf{U}^*_{ij} = \mathbf{U}\delta_{ij}$; and let $\mathbf{U}_*$ denote the composite matrix dual to $\mathbf{U}^*$ whose components are the scalar matrices $\mathbf{U}^{rs}_* = U_{rs}\mathbf{1}$. Let composite matrices $(\mathbf{U}^{-1})^*$ and its dual $(\mathbf{U}^{-1})_*$ be similarly formed from the inverse $\mathbf{U}^{-1}$ of $\mathbf{U}$. Then $\mathbf{B}^* = \mathbf{U}^*\mathbf{A}^*(\mathbf{U}^{-1})^*$ is a composite matrix whose components are all triangular matrices, and hence its dual, which, by Theorem I, is

$\mathbf{B}_* = \mathbf{U}_* \mathbf{A}_* (\mathbf{U}^{-1})_*$, is a triangular composite matrix whose μth diagonal component is the characteristic dual component $\boldsymbol{\alpha}_*^\mu$ of $\mathbf{A}^*$. Now the ground matrices $\mathbf{a}^*$ of $\mathbf{A}^*$ and $\mathbf{b}^*$ of $\mathbf{B}^*$ have the same characteristic values. For, by Theorem II, $(\mathbf{U}^{-1})^* = \mathbf{U}^{*-1}$, and also the ground matrix of $\mathbf{U}^{*-1}$ is the inverse $\mathbf{u}^{*-1}$ of the ground matrix $\mathbf{u}^*$ of $\mathbf{U}^*$, which shows, by Theorem I, that $\mathbf{b}^* = \mathbf{u}^*\mathbf{a}^*\mathbf{u}^{*-1}$, i.e. that $\mathbf{a}^*$ and $\mathbf{b}^*$ are similar matrices. Moreover, the ground matrix $\mathbf{b}_*$ of $\mathbf{B}_*$ has the same characteristic values as $\mathbf{b}^*$. But the characteristic values of the ground matrix of a triangular composite matrix are the characteristic values of its diagonal components, since the characteristic polynomial of the ground matrix splits into the product of the characteristic polynomials of the diagonal components. Thus the characteristic values of $\mathbf{b}_*$, or equivalently the characteristic values of $\mathbf{b}^*$, and hence those of $\mathbf{a}^*$, are the characteristic values of the diagonal components of $\mathbf{B}_*$, that is to say of the matrices $\boldsymbol{\alpha}_*^\mu$. Thus the theorem is proved.

For the next theorem, the following lemma will be required.

LEMMA. *If* $|\mathbf{x}\ \mathbf{y}|$ *denotes a determinant partitioned in its columns, and* $\mathbf{a}$ *is a square matrix with the same number of columns as* $\mathbf{x}$, *then*

$$|\mathbf{xa}\ \mathbf{y}| = |\mathbf{a}|\,|\mathbf{x}\ \mathbf{y}|.$$

When $\mathbf{x}$ is replaced by $\mathbf{xa}$, respective minors on the columns of $\mathbf{x}$ are multiplied by the determinant of $\mathbf{a}$; and since, by Laplace's theorem, a determinant is a linear function of the minors on any set of its columns, it follows that $|\mathbf{x}\ \mathbf{y}|$ is then replaced by $|\mathbf{a}|\,|\mathbf{x}\ \mathbf{y}|$.

THEOREM V. *The double determinant of a commutative composite matrix is equal to its ground determinant.*

The proof is by induction on the order n of the composite matrix. As the initial proposition, we have the case $n = 1$; which is verified, since the determinant and the ground matrix of a composite matrix with a single component are both equal to that component. Our hypothesis for induction is the validity of the theorem for commutative composite matrices of order $n-1$. We have a commutative composite matrix $\mathbf{A}$ of order n, with ground matrix $\mathbf{a}$; and we suppose that $n > 1$. It is required to prove that $\|\mathbf{A}\| = |\mathbf{a}|$, given the commutativity of $\mathbf{A}$, and the inductive hypothesis.

By definition,

$$|\mathbf{A}| = \mathbf{A}_{11}|\mathbf{B}_1| - \mathbf{A}_{12}|\mathbf{B}_2| + \ldots + (-1)^{n-1}\mathbf{A}_{1n}|\mathbf{B}_n|,$$

this being the expansion of the determinant of $\mathbf{A}$ on the components in its first row, where $\mathbf{B}_i$ denotes the composite matrix of order $n-1$

formed from all the rows of $\mathbf{A}$, with the exception of the first, and all the columns of $\mathbf{A}$, with the exception of the ith. If $\mathbf{b}_i$ denotes the ground matrix of $\mathbf{B}_i$, then, by the inductive hypothesis, we have $\|\mathbf{B}_i\| = |\mathbf{b}_i|$, since $\mathbf{B}_i$ is commutative and of order $n-1$. But, by the lemma, together with the theorem on the multiplication of partitioned matrices, we have

$$\|\mathbf{B}_1\|\,|\mathbf{a}| = \begin{vmatrix} \mathbf{A}_{11}|\mathbf{B}_1| & \mathbf{A}_{12} & . & . & . & \mathbf{A}_{1n} \\ . & . & . & . & . & . \\ \mathbf{A}_{n1}|\mathbf{B}_1| & \mathbf{A}_{n2} & . & . & . & \mathbf{A}_{nn} \end{vmatrix};$$

and then, by elementary operations on the columns, which leave the value of the determinant unchanged, this expression becomes

$$\begin{vmatrix} \mathbf{A}_{11}|\mathbf{B}_1|-\ldots+(-1)^{n-1}\mathbf{A}_{1n}|\mathbf{B}_n| & \mathbf{A}_{12} & . & . & . & \mathbf{A}_{1n} \\ \mathbf{A}_{21}|\mathbf{B}_1|-\ldots+(-1)^{n-1}\mathbf{A}_{2n}|\mathbf{B}_n| & \mathbf{A}_{22} & . & . & . & \mathbf{A}_{2n} \\ . \quad . & . & . & . & . & . \\ \mathbf{A}_{n1}|\mathbf{B}_1|-\ldots+(-1)^{n-1}\mathbf{A}_{nn}|\mathbf{B}_n| & \mathbf{A}_{n2} & . & . & . & \mathbf{A}_{nn} \end{vmatrix}.$$

The commutation conditions forming the hypothesis in the theorem now give that all the components in the first column here are null, with the exception of the first, which is $|\mathbf{A}|$; for they are all expansions of the determinants of composite matrices which have two of their rows identical, and thus vanish in consequence of the commutation relations, with the exception of the first, which represents an expansion of $|\mathbf{A}|$. Hence the expression for the determinant reduces to

$$\left|\begin{array}{c|ccc} |\mathbf{A}| & \mathbf{A}_{12} & . \;\; . \;\; . & \mathbf{A}_{1n} \\ \hline \mathbf{0} & \multicolumn{3}{c}{\mathbf{b}_1} \end{array}\right| = |\mathbf{b}_1|\,\|\mathbf{A}\|.$$

Now, since $|\mathbf{b}_1| = \|\mathbf{B}_1\|$, it is shown that

$$|\mathbf{b}_1|\,\|\mathbf{A}\| = |\mathbf{b}_1|\,|\mathbf{a}|.$$

Thus we have $\|\mathbf{A}\| = |\mathbf{a}|$, provided that $|\mathbf{b}_1| \neq 0$. But the inequality here is irrelevant to this conclusion, by the principle of the irrelevance of algebraic inequalities [(**15**) 4]. Accordingly, we have proved $\|\mathbf{A}\| = |\mathbf{a}|$, as required; and the induction is now complete.

For an example, let $\mathbf{a}$ be any simple square matrix of order n with elements a_{ij}. Then the elements

$$C^{rs}_{ij} = a_{rs}\delta_{ij} - a_{ij}\delta_{rs}$$

form the ground elements of a doubly commutative dual pair of composite matrices $\mathbf{C}^*$ and $\mathbf{C}_*$, of both order and degree n, with components $\mathbf{C}^*_{ij} = \mathbf{a}\delta_{ij} - a_{ij}\,\mathbf{1}^*$ and $\mathbf{C}^{rs}_* = a_{rs}\,\mathbf{1}_* - \mathbf{a}\delta_{rs}$, which are such that $\mathbf{C}_* = -\mathbf{C}^*$.

The Cayley–Hamilton theorem states that any matrix satisfies its characteristic equation, and an equivalent of this statement is $|\mathbf{C}^*| = \mathbf{0}^*$. Let $\mathbf{c}^*$ denote the ground matrix of $\mathbf{C}^*$. We may call $\mathbf{C}^*$ the characteristic composite matrix of $\mathbf{a}$. Then as a consequence of the present theorem we have: *The ground matrix of the characteristic composite matrix of any simple matrix is singular.*† For we have

$$|\mathbf{c}^*| = \|\mathbf{C}^*\| = |\mathbf{0}^*| = 0.$$

Now let $\mathbf{a}$, $\mathbf{b}$ be any square matrices, of order m, n respectively. Then the elements $P_{ij}^{rs} = a_{rs} b_{ij}$ form the ground elements of dual commutative composite matrices $\mathbf{P}^*$, $\mathbf{P}_*$ with components $\mathbf{P}_{ij}^* = \mathbf{a} b_{ij}$, $\mathbf{P}_*^{rs} = a_{rs}\mathbf{b}$ and ground matrices $\mathbf{p}^*$, $\mathbf{p}_*$, each of which represents the direct product of $\mathbf{a}$ and $\mathbf{b}$. Now we have $\mathbf{P}^* = \mathbf{a}\mathbf{b}^{**}$, where $\mathbf{b}^{**}$ denotes the composite matrix with the scalar matrix $b_{ij}\mathbf{1}^*$ as its (i,j)th component. Hence

$$|\mathbf{P}^*| = \mathbf{a}^n |\mathbf{b}| \mathbf{1}^* \quad \text{and} \quad \|\mathbf{P}^*\| = |\mathbf{a}|^n |\mathbf{b}|^m.$$

But, by the present theorem, we have $|\mathbf{p}^*| = \|\mathbf{P}^*\|$, and thus we deduce: *the determinant of the direct product of matrices* $\mathbf{a}$, $\mathbf{b}$ *of order* m, n *is* $|\mathbf{a}|^n |\mathbf{b}|^m$. We may also use the present example to illustrate Theorem IV, on the ground characteristic values of a commutative composite matrix. It shows here that the characteristic values of $\mathbf{p}^*$ are the characteristic values of the matrices of the form $\alpha\mathbf{b}$, where α is a characteristic value of $\mathbf{a}$, say of multiplicity m_α; from which it follows immediately that they are the products $\alpha\beta$, where β is a characteristic value of $\mathbf{b}$, say of multiplicity m_β, and that the corresponding multiplicities are given by $m_{\alpha\beta} = m_\alpha m_\beta$. Accordingly, *the characteristic values of a direct product of matrices are the products of the respective characteristic values of the matrices.*

To state how the ground characteristic polynomial of a composite matrix factorizes into a product of characteristic polynomials of matrices, when its dual components commute, we have that, if $\mathbf{A}$ is a composite matrix whose dual components commute, $\mathbf{a}$ its ground matrix, and $\boldsymbol{\alpha}$ its characteristic components, with multiplicities $m_{\boldsymbol{\alpha}}$ respectively, then

$$|\lambda\mathbf{1} - \mathbf{a}| = \prod_{\boldsymbol{\alpha}} |\lambda\mathbf{1} - \boldsymbol{\alpha}|^{m_{\boldsymbol{\alpha}}}.$$

This is an equivalent of Theorem IV. When the components of $\mathbf{A}$ also commute, Theorem V now gives

$$\|\lambda\mathbf{1} - \mathbf{A}\| = \prod_{\boldsymbol{\alpha}} |\lambda\mathbf{1} - \boldsymbol{\alpha}|^{m_{\boldsymbol{\alpha}}}.$$

† This result, which is included for an illustration, was mentioned to me, with a simple proof, by Dr. G. Szekeres.

From here we have the suggestion that the more comprehensive relation

$$|\lambda \mathbf{1} - \mathbf{A}| = \prod_{\mathbf{a}} (\lambda \mathbf{1} - \boldsymbol{\alpha})^{m_{\mathbf{a}}}$$

might hold; which is in fact the case, as will appear in due course.

The characteristic values of the ground matrices $\mathbf{a}^*$, $\mathbf{a}_*$ for any dual composite pair $\mathbf{A}^*$, $\mathbf{A}_*$ are identical, and define the ground characteristic values for that dual pair. If $\mathbf{A}^*$, $\mathbf{A}_*$ are commutative, let $\mathbf{X}^*_\mu$ and $\mathbf{X}^\nu_*$ be the component latent vectors, that is to say common latent vectors of the components of $\mathbf{A}^*$ and $\mathbf{A}_*$ respectively, as have been defined. They are such that

$$\mathbf{A}^*_{ij}\,\mathbf{X}^*_\mu = \mathbf{X}^*_\mu\,\alpha^\mu_{ij}, \qquad \mathbf{A}^{rs}_*\,\mathbf{X}^\nu_* = \mathbf{X}^\nu_*\,\alpha^{rs}_*.$$

As in Theorem III, we have

$$\sum_s \mathbf{A}^{rs}_* X^*_{\mu,s} = \boldsymbol{\alpha}^\mu_*\, X^*_{\mu,r}, \qquad \sum_j \mathbf{A}^*_{ij}\, X^{\nu,j}_* = \boldsymbol{\alpha}^*_\nu\, X^{\nu,i}_*;$$

from which it may be deduced, by the theorem on the characteristic values of functions of commutative matrices in § 2, that the characteristic values of $\boldsymbol{\alpha}^\mu_*$ and $\boldsymbol{\alpha}^*_\nu$ are the values $\alpha^\mu_{*\nu}$ $(\nu = 1,\ldots,n)$ and $\alpha^{*\mu}_\nu$ $(\mu = 1,\ldots,m)$, respectively, which satisfy the relations

$$\sum_s \alpha^{rs}_\nu\, X^*_{\mu,s} = \alpha^\mu_{*\nu}\, X^*_{\mu,r}, \qquad \sum_j \alpha^\mu_{ij}\, X^{\nu,j}_* = \alpha^{*\mu}_\nu\, X^{\nu,i}_*.$$

We further have that the latent vectors corresponding to these characteristic values are $\mathbf{X}^\nu_*$ and $\mathbf{X}^*_\mu$ respectively: that is to say

$$\boldsymbol{\alpha}^\mu_*\,\mathbf{X}^\nu_* = \mathbf{X}^\nu_*\,\alpha^\mu_{*\nu}, \qquad \boldsymbol{\alpha}^*\mathbf{X}^*_\mu = \mathbf{X}^*_\mu\,\alpha^{*\mu}_\nu;$$

that is,

$$\sum_j \alpha^\mu\, X^{\nu,j}_* = \alpha^\mu_{*\nu}\, X^{\nu,i}_*, \qquad \sum_s \alpha^{rs}_\nu\, X^*_{\mu,s} = X^*_{\mu,r}\,\alpha^{*\mu}_\nu.$$

By comparison with the previous relation, this gives

$$\alpha^\mu_{*\nu} = \alpha^{*\mu}_\nu.$$

It can be seen† that the arrays $\alpha^\mu_{*\nu}$ $(\nu = 1,\ldots,\ n;\ \mu = 1,\ldots,\ m)$ and $\alpha^{*\mu}_\nu$ $(\mu = 1,\ldots,\ m;\ \nu = 1,\ldots,\ n)$ display the sets of characteristic sets of the matrix sets $\boldsymbol{\alpha}^\mu_*$ $(\mu = 1,\ldots,\ m)$ and $\boldsymbol{\alpha}^*_\nu$ $(\nu = 1,\ldots,\ n)$, these being commutative, as stated in Theorem III, Corollary II; and they define the characteristic arrays of the commutative dual pair $\mathbf{A}^*$, $\mathbf{A}_*$, for which we have the following result:

THEOREM VI. *The characteristic arrays of each of a dual pair of composite matrices are the transposes of each other, and their complementary sections are the characteristic sets of the commutative matrix sets formed by the complementary systems of characteristic components.*

† This is by consequence of the proposition that the characteristic sets of any set $\{\mathbf{F} = f_\iota(\mathbf{s})\}_{\iota \in I}$ of rational functions of some commutative matrix set $\mathbf{s}$ are the sets $\Phi_\sigma = \{f_\iota(\sigma)\}_{\iota \in I}$, where σ ranges over the characteristic sets of $\mathbf{s}$.

Now we may write

$$\alpha^{\mu}_{*\nu} = \alpha^{\mu}_{\nu} = \alpha^{*\mu}_{\nu}$$

and say α^{μ}_{ν} defines *the* characteristic array for the dual pair, and give priority to either index according as the pair is viewed from one member or the other, the sections of the array by the index ν forming the characteristic sets for the set of matrices $\boldsymbol{\alpha}^{\mu}_{*}$, and the sections by μ, for the matrices $\boldsymbol{\alpha}^{*}_{\nu}$. As a consequence of Theorem IV, we have:

THEOREM VII. *The ground characteristic values of a doubly commutative dual pair of composite matrices are the elements of its characteristic array.*

Now consider again the sets of relations

$$\sum_{s} \mathbf{A}^{rs}_{*} X^{*}_{\mu,s} = \boldsymbol{\alpha}^{\mu}_{*} X^{*}_{\mu,r}, \qquad \sum_{j} \mathbf{A}^{*}_{ij} X^{*v,j}_{*} = \boldsymbol{\alpha}^{*}_{\nu} X^{*v,i}_{*},$$

obtained from Theorem III. Since all the components $\mathbf{A}^{rs}_{*}$ commute, for each μ we may eliminate the scalar quantities $X^{*}_{\mu,r}$, not all of which are zero, from the first set, and thus obtain the eliminant

$$|\boldsymbol{\alpha}^{\mu}_{*}\mathbf{1}_{**} - \mathbf{A}_{*}| = \mathbf{0}_{*}.$$

Similarly there may be obtained the eliminant $|\boldsymbol{\alpha}^{*}_{\nu}\mathbf{1}^{**} - \mathbf{A}^{*}| = \mathbf{0}^{*}$ from the second set for each ν. Let $\boldsymbol{\alpha}^{*}$, $\boldsymbol{\alpha}_{*}$ denote indeterminates in the algebras generated by the components of $\mathbf{A}^{*}$, $\mathbf{A}_{*}$ respectively, and define

$$\mathbf{C}^{*}(\boldsymbol{\alpha}^{*}) = |\boldsymbol{\alpha}^{*}\mathbf{1}^{**} - \mathbf{A}^{*}|, \qquad \mathbf{C}_{*}(\boldsymbol{\alpha}_{*}) = |\boldsymbol{\alpha}_{*}\mathbf{1}_{**} - \mathbf{A}_{*}|,$$

which we have called the *composite characteristic* polynomials of $\mathbf{A}^{*}$, $\mathbf{A}_{*}$. We have shown that

$$\mathbf{C}^{*}(\boldsymbol{\alpha}^{*}_{\nu}) = \mathbf{0}^{*}, \qquad \mathbf{C}_{*}(\boldsymbol{\alpha}^{\mu}_{*}) = \mathbf{0}$$

and we express this in the following theorem:

THEOREM VIII. *The characteristic components of a dual pair of commutative composite matrices are roots of the respective composite characteristic polynomials.*

The Cayley–Hamilton theorem, that any matrix satisfies its characteristic equation, supplies a system of algebraic identities in the elements of a matrix, and these identities remain valid when the components of any commutative composite matrix are substituted. Accordingly, we have

$$\mathbf{C}^{*}(\mathbf{A}^{*}) = \mathbf{0}^{**}, \qquad \mathbf{C}_{*}(\mathbf{A}_{*}) = \mathbf{0}_{**},$$

i.e. generally:

THEOREM IX. *A commutative composite matrix satisfies its composite characteristic equation.*

The next theorem exhibits the composite characteristic polynomial completely split into a product of factors, and shows that the characteristic components form a complete set of roots, with correspondence for the multiplicities.

THEOREM X. *If* $\mathbf{A}^*$, $\mathbf{A}_*$ *are dual commutative composite matrices, and* $\boldsymbol{\alpha}_\nu^*$, $\boldsymbol{\alpha}_*^\mu$ *the characteristic components, then*

$$|\boldsymbol{\alpha}^*\mathbf{1}^{**}-\mathbf{A}^*| = \prod_{\nu=1}^{n}(\boldsymbol{\alpha}^*-\boldsymbol{\alpha}_\nu^*), \qquad |\boldsymbol{\alpha}_*\mathbf{1}_{**}-\mathbf{A}_*| = \prod_{\mu=1}^{m}(\boldsymbol{\alpha}_*-\boldsymbol{\alpha}_*^\mu).$$

Since the components of $\mathbf{A}^*$ commute, there exists, by the last transformation theorem stated in § 2, a regular matrix $\mathbf{T}$ by which these components $\mathbf{A}_\nu^*$ are transformed into direct sums

$$\mathbf{B}_{ij}^* = \mathbf{T}^{-1}\mathbf{A}_{ij}^*\mathbf{T} = \bigoplus_k \mathbf{A}_{ij}^{*k}$$

of triangular matrices $\mathbf{A}_{ij}^{*k}$, with equal diagonal elements, such that the μth diagonal element of $\mathbf{B}_{ij}^*$ is the (i,j)th element α_{ij}^μ of the characteristic dual component $\boldsymbol{\alpha}_*^\mu$ of $\mathbf{A}^*$. The composite matrix $\mathbf{B}^*$, whose components are these transforms, is given by $\mathbf{B}^* = (\mathbf{T}^{-1})^*\mathbf{A}^*\mathbf{T}^*$, where $\mathbf{T}^*$ has components $\mathbf{T}_{ij}^* = \mathbf{T}\delta_{ij}$, and $(\mathbf{T}^{-1})^*$ is determined in the same way from $\mathbf{T}^{-1}$. Evidently $(\mathbf{T}^{-1})^* = \mathbf{T}^{*-1}$; and, by Theorem II, we have

$$(\mathbf{T}^{-1})_* = \mathbf{T}_*^{-1}$$

for the dual of $(\mathbf{T}^{-1})^*$. Thus we have $\mathbf{B}_* = \mathbf{T}_*^{-1}\mathbf{A}_*\mathbf{T}_*$ for the dual of $\mathbf{B}^*$, by Theorem I. From the form of the components of $\mathbf{B}^*$, we see that $\mathbf{B}_*$ is a triangular composite matrix whose μth diagonal component is $\boldsymbol{\alpha}_*^\mu$. Hence we deduce that

$$|\boldsymbol{\alpha}_*\mathbf{1}_{**}-\mathbf{B}_*| = \prod_{\mu=1}^{m}(\boldsymbol{\alpha}_*-\boldsymbol{\alpha}_*^\mu).$$

But $\mathbf{B}_* = \mathbf{T}_*^{-1}\mathbf{A}_*\mathbf{T}_*$; and, since the multiplication theorem for simple determinants is valid for composite determinants, it follows directly that

$$|\boldsymbol{\alpha}_*\mathbf{1}_{**}-\mathbf{A}_*| = |\boldsymbol{\alpha}_*\mathbf{1}_{**}-\mathbf{B}_*|.$$

One part of the theorem is thus proved, and the other part is its dual equivalent.

An immediate consequence of Theorems IX and X is the following system of identities between the components and the characteristic components of a doubly commutative composite matrix.

THEOREM XI. *If* $\mathbf{A}^*$, $\mathbf{A}_*$ *are dual commutative composite matrices, and* $\boldsymbol{\alpha}_\nu^*$, $\boldsymbol{\alpha}_*^\mu$ *are the characteristic components, then*

$$\prod_{\nu=1}^{n}(\mathbf{A}^*-\boldsymbol{\alpha}_\nu^*\mathbf{1}^{**}) = \mathbf{0}^{**}, \qquad \prod_{\mu=1}^{m}(\mathbf{A}_*-\boldsymbol{\alpha}_*^\mu\mathbf{1}_{**}) = \mathbf{0}_{**}.$$

We may note that, given any algebras K, L of matrices of order n, m respectively, we can form an algebra M^* of composite matrices, of order n and degree m, any member $\mathbf{X}^*$ of which has components of the form $\mathbf{X}^*_{ij} = \mathbf{a}u_{ij}+\mathbf{b}v_{ij}+\ldots$, where $\mathbf{a}$, $\mathbf{b}$,... are matrices which belong to K and u_{ij}, v_{ij},... are the elements of matrices $\mathbf{u}$, $\mathbf{v}$,... which belong to L; and the corresponding dual components are $\mathbf{X}^{rs}_* = a_{rs}\mathbf{u}+b_{rs}\mathbf{v}+\ldots$. There is also the algebra M_* of the duals, and the algebra M of the ground matrices, the three algebras so defined being isomorphic and representing the direct product of the algebras represented by K and L. If K is a commutative algebra, then the composite matrices forming M^* have commutative components, and, if, further, L is commutative, then the dual components are also commutative; and, moreover, M^* will then be a commutative algebra. Thus from any pair of commutative matrix algebras a commutative algebra of doubly commutative composite matrices is formed. It may be remarked that a relation analogous to the relation of latent association which holds between characteristic values of commutative matrices can be defined between characteristic components of the members in such an algebra, and on the basis of this definition it is possible to extend some of our results. Now let K and L be of the same order; then for the arbitrary member $\mathbf{X}^*$ of M^* there is defined the matrix $\mathbf{x} = \mathbf{ua}+\mathbf{vb}+\ldots$; also, if L is commutative, then the determinant $|\mathbf{X}^*|$ is defined. The next theorem, which is due to Phillips (7), shows that $|\mathbf{X}^*| = \mathbf{0}$ if $\mathbf{x} = \mathbf{0}$.

THEOREM XII. *If* $\mathbf{x} = \mathbf{ua}+\mathbf{vb}+\ldots$, *where* $\mathbf{a}$, $\mathbf{b}$,... *commute, and if* $\mathbf{X}_{ij} = u_{ij}\mathbf{a}+v_{ij}\mathbf{b}+\ldots$, *then* $|\mathbf{X}| = \mathbf{0}$ *if* $\mathbf{x} = \mathbf{0}$.

We have $\mathbf{x} = \sum \mathbf{e}_{ij}\mathbf{X}_{ij}$, where $\mathbf{e}_{ij}$ has the (i,j)th of its elements unity and the rest zero; and thus $\mathbf{e}_{rs}\mathbf{x} = \sum_j \mathbf{e}_{rj}\mathbf{X}_{sj}$. Let $\mathbf{Y}$ denote the adjugate composite matrix of $\mathbf{X}$ [(16) 166, footnote]. Hence, if $\mathbf{x} = 0$, we obtain

$$\mathbf{0} = \sum_s \Big(\sum_j \mathbf{e}_{rj}\mathbf{X}_{sj}\Big)\mathbf{Y}_{ts} = \sum_j \mathbf{e}_{rj}\delta_{ij}|\mathbf{X}| = \mathbf{e}_{rt}|\mathbf{X}|,$$

for each t and r; which implies $|\mathbf{X}| = \mathbf{0}$.

The Cayley–Hamilton theorem can be seen as a special case of this theorem. For we have $\mathbf{x} = \mathbf{1a}-\mathbf{a1} = \mathbf{0}$, for any matrix $\mathbf{a}$. Hence, if $\mathbf{X}_{ij} = \delta_{ij}\mathbf{a}-a_{ij}\mathbf{1}$, then $|\mathbf{X}| = 0$, which is to say that, if $C(\lambda) = |\lambda\mathbf{1}-\mathbf{a}|$, then $C(\mathbf{a}) = \mathbf{0}$.

A connexion can be seen between the subject of this paper and the scheme which Dirac (17) has given, by which certain dynamical systems can be viewed as composite systems formed from two separate dynamical

systems, with the vectors defining the states, and the linear operators defining the dynamical variables formed correspondingly, with commuting sets of observables for the separate systems giving rise to commuting sets of observables for the composite systems.

I wish to express my thanks to Dr. W. L Ferrar for the interest he has taken in the carrying out of this work.

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NOTE ON A PREVIOUS PAPER

By R. WILSON (*Swansea*)

[Received 23 August 1953]

In a recent paper (**2**)† determinantal criteria were derived concerning the properties of integral functions of finite order ρ and mean type h, defined by the series

$$F(z) = \sum_0^\infty a_n z^n \equiv \sum_0^\infty \frac{c_n z^n}{\Gamma(n\sigma+\sigma)}, \tag{1}$$

where $\sigma = \rho^{-1}$. These arise from coefficient properties of the generalized Laplace transform of (1)

$$f(z) = \sum_0^\infty \Gamma(n\sigma+\sigma)a_n z^{-n-1} \equiv \sum_0^\infty c_n z^{-n-1} \tag{2}$$

when it possesses dominant singularities which are polar. The function (2) is regular outside the circle $|z| = h^\sigma$ and singular at one point of the circumference of this circle at least.

The generalized inverse transform of

$$(z-\alpha)^{-1} = \sum_0^\infty \alpha^n z^{-n-1}$$

is

$$\sum_0^\infty \frac{\alpha^n z^n}{\Gamma(n\sigma+\sigma)} \equiv E_\rho(\alpha z),$$

while that of

$$(z-\alpha)^{-m-1} = \frac{1}{m!}\sum_m^\infty n(n-1)\ldots(n-m+1)\alpha^{n-m}z^{-n-1}$$

is

$$\frac{1}{m!}\frac{\partial^m}{\partial\alpha^m}E_\rho(\alpha z).$$

In the case of $\rho = 1 = \sigma$ the corresponding theory is made much easier by the fact that the ordinary inverse transform of a multiple pole

$$\sum_1^m \frac{A_\mu}{(z-\alpha)^\mu} \tag{3}$$

takes the simple form

$$\sum_1^m \left(\frac{A_\mu}{(\mu-1)!}z^{\mu-1}\right)e^{\alpha z}$$

† There are a number of minor errors in this paper. In (**4**), (**13**), and (**14**), and at the end of § 4, $A_{\mu\nu}$ should be replaced by $A_{\mu\nu}/(\mu-1)!$. In the line following (16), h^ρ should read h^σ. Finally the $|\mathfrak{D}_{n,p}|$ arising four times in the statement of Theorem 3 should each be raised to the power ρ/n and not as shown.

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and it was pointed out that, in the general case when $\rho \neq 1$, such polynomial multipliers do not arise in association with the $E_\rho(\alpha z)$. For that reason Theorem 3 of (**2**) was restricted to the case in which the dominant poles of $f(z)$ are simple.

However, since the generalized transform of (3) is

$$\sum_1^m \left\{ \frac{A_\mu}{(\mu-1)!} \frac{\partial^{\mu-1}}{\partial \alpha^{\mu-1}} E_\rho(\alpha z) \right\},$$

it is clear that Theorem 3 of (**2**) can be stated in the more general form corresponding to the case in which the dominant poles of $f(z)$ are multiple.

First, however, I give a correlative of a theorem due to Kronecker [(**1**) 321], in which $D_{n,s}$ denotes the Hankel determinant

$$|c_{n+\mu+\nu}| \quad (\mu, \nu = 0, 1, 2, \ldots, s).$$

Kronecker's theorem may be stated in the form given below.

THEOREM 1*. *The necessary and sufficient condition that*

$$f(z) = \sum c_n z^{-n-1}$$

shall be a rational function is that an integer m exists such that $D_{0,s} = 0$ for $s \geqslant m$.

The use of the methods developed in (**2**) leads to Theorem 2*, in which $\mathfrak{D}_{n,s}$ denotes the determinant formed from $D_{n,s}$ by replacing each $c_{n+\nu}$ by $\Gamma[(n+\nu+1)\sigma]a_{n+\nu}$.

THEOREM 2*. *The necessary and sufficient condition that an integral function $\sum a_n z^n$ of order ρ shall consist of a finite number of linear sums of the form $E_\rho(\alpha_\nu z)$ and its derivatives with respect to α_ν, is that an integer m exists such that $\mathfrak{D}_{0,s} = 0$ for $s \geqslant m$.*

Note that the type h of the integral function is $\max|\alpha_\nu|^\rho$.

A more general form of Theorem 3 of (**2**) now follows.

THEOREM 3.* *Necessary and sufficient conditions that an integral function of order ρ and mean type, $F(z) = \sum a_n z^n$, shall have dominant elements of the form $E_\rho(\alpha_\nu z)$ and its derivates with respect to α_ν, each of type h_1, secondary elements of a similar kind of type h_2, and so on, are that*

$$\limsup_{n\to\infty} |\mathfrak{D}_{n,\nu}|^{\rho/n} = h_1^{\nu+1} \quad (\nu = 0, 1, 2, \ldots, p_1-2),$$

$$\lim_{n\to\infty} |\mathfrak{D}_{n,p_1-1}|^{\rho/n} = h_1^{p_1},$$

$$\limsup_{n\to\infty} |\mathfrak{D}_{n,p_1+\nu}|^{\rho/n} = h_1^{p_1} h_2^{\nu+1} \quad (\nu = 0, 1, 2, \ldots, p_2-2),$$

$$\lim_{n\to\infty} |\mathfrak{D}_{n,p_1+p_2-1}|^{\rho/n} = h_1^{p_1} h_2^{p_2},$$

and so on.

Note that the exponents α_ν of type h_1 (for which $h_1 = |\alpha_\nu|^\rho$) are given by the zeros of the polynomial

$$\lim_{n\to\infty} \mathfrak{R}_{n,p_1}(z)/\mathfrak{D}_{n,p_1-1}$$

defined in (**2**), and the exponents of type h_2 by the *additional* zeros of

$$\lim_{n\to\infty} \mathfrak{R}_{n,p_1+p_2}(z)/\mathfrak{D}_{n,p_1+p_2-1},$$

and so on. The multiplicity of any zero α_ν determines the number of derivatives of the form $(\partial/\partial\alpha_\nu)^\mu E_\rho(\alpha_\nu z)$ which occur in $F(z)$, this number being one less than the multiplicity of the zero concerned.

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ON THE REDUCIBILITY OF POLYNOMIALS OVER A FINITE FIELD

By M. C. R. BUTLER (*Melbourne*)

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1. Introduction

THE aim of this paper is to establish a finite, non-tentative method of determining the number of different irreducible factors of any polynomial over a finite (or Galois) field. This number will be shown to be equal to the number of linearly independent solutions of a certain system of linear equations; the coefficients in this system of equations will appear as rational integral functions of the coefficients of the given polynomial.

This 'linearization' of the reducibility problem makes essential use of the cyclicity of the multiplication group of a finite field: that is, of the property that, if $\alpha \neq 0$ is an element of the finite field $\mathrm{GF}(p^n)$, then it satisfies the relation

$$\alpha^{p^n-1} = 1.$$

As a consequence, each element α of $\mathrm{GF}(p^n)$ satisfies

$$\alpha^{p^n} = \alpha.$$

If $f(x)$ is a separable polynomial with coefficients in $\mathrm{GF}(p^n)$, it will appear that a study of the congruence

$$Z^{p^n} \equiv Z \pmod{f(x)}$$

leads to information about the factors of $f(x)$ in $\mathrm{GF}(p^n)$, but before showing this we need some general results from the theory of rings.

2. The factorization of the remainder-class ring

Let K denote a field, x an indeterminate over K, and $f(x)$ a polynomial of degree m with coefficients in K. We shall be concerned with the ring R of remainder classes, modulo $f(x)$, of the ring of all polynomials in x with coefficients in K. It is well known that each remainder class may be represented uniquely by a polynomial of degree not exceeding $m-1$ with coefficients in K, whilst, conversely, each such polynomial represents exactly one remainder class. For this reason, elements of R will often be called 'polynomials', and all calculations with elements of R will be carried out by means of their polynomial representatives. Assume that

$$f(x) = f_1(x)^{k_1} \ldots f_r(x)^{k_r} \quad (k_1, \ldots, k_r \geqslant 1) \tag{1}$$

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is the factorization of $f(x)$ into the powers of polynomials $f_1(x),..., f_r(x)$, each of which has coefficients in K and is irreducible over K. It may then be shown (3) that R is the direct sum of exactly r rings R_i:

$$R = R_1 \oplus R_2 \oplus ... \oplus R_r,$$

where $$R_i = K[x]/\{f_i(x)^{k_i}\} \quad (i = 1, 2,..., r).$$

This representation of R may be based (1) on the existence of r polynomials

$$\eta_1(x),\ \eta_2(x),...,\ \eta_r(x)$$

which have the following characteristic properties:

$$\eta_1(x)+...+\eta_r(x) \equiv 1 \ (\mathrm{mod} f(x)), \tag{2}$$

$$\eta_i^2(x) \equiv \eta_i(x) \ (\mathrm{mod} f(x)) \quad (i = 1, 2,..., r), \tag{3}$$

$$\eta_i(x)\eta_j(x) \equiv 0 \ (\mathrm{mod} f(x)) \quad (i \neq j;\ i, j = 1, 2,..., r). \tag{4}$$

With suitable numbering of the $\eta_i(x)$,

$$\eta_i(x) \equiv 1 \ (\mathrm{mod} f_i(x)^{k_i}) \quad (i = 1, 2,..., r), \tag{5}$$

$$\eta_i(x) \equiv 0 \ (\mathrm{mod} f_j(x)^{k_j}) \quad (i \neq j;\ i, j = 1, 2,..., r). \tag{6}$$

Finally, it can be proved that each element $\zeta(x)$ of R has a unique representation in the form

$$\zeta(x) \equiv \zeta_1(x)\eta_1(x)+...+\zeta_r(x)\eta_r(x) \quad (\mathrm{mod} f(x)), \tag{7}$$

where $$\zeta_i(x) \equiv \zeta(x) \ (\mathrm{mod} f_i(x)^{k_i}) \quad (i = 1, 2,..., r). \tag{8}$$

3. The congruence $Z^{p^n} \equiv Z \bmod f(x)$

We now impose the restriction that K be a finite field $\mathrm{GF}(p^n)$. Then

$$R = \mathrm{GF}(p^n)[x]/\{f(x)\}.$$

THEOREM 1. *Each solution in R of the congruence*

$$Z^{p^n} \equiv Z \ (\mathrm{mod} f(x)) \tag{9}$$

can be expressed in the form

$$\alpha_1 \eta_1(x)+...+\alpha_r \eta_r(x) \ (\mathrm{mod} f(x)), \tag{10}$$

where $\alpha_1,..., \alpha_r$ are in $\mathrm{GF}(p^n)$ and $\eta_1(x),..., \eta_r(x)$ are the polynomials introduced above. Conversely, every expression of the form (10) *is a solution of* (9).

The set of all solutions in R of (9) *forms a subring S of R.*

Proof. The relation (3) shows that $\eta_1(x),..., \eta_r(x)$ are each solutions of (9), whilst the cyclicity of the field $\mathrm{GF}(p^n)$ ensures that each of its elements is also a solution.

Let Z_1, Z_2 be two solutions of (9). Then

$$(Z_1 Z_2)^{p^n} = Z_1^{p^n} Z_2^{p^n}$$
$$\equiv Z_1 Z_2 \pmod{f(x)}.$$

Also
$$(Z_1+Z_2)^{p^n} = Z_1^{p^n}+Z_2^{p^n} \quad \text{over GF}(p^n)$$
$$\equiv Z_1+Z_2 \pmod{f(x)},$$

so that $Z_1 Z_2$ and Z_1+Z_2 are also solutions of (9). This proves that the set of all solutions of (9) is a ring; in particular, each expression of the form (10) is a solution of (9).

Now let $\zeta(x)$ be any solution in R of (9). We express it in its form (7),

$$\zeta(x) \equiv \zeta_1(x)\eta_1(x)+\ldots+\zeta_r(x)\eta_r(x), \tag{11}$$

where $\zeta_i(x) \equiv \zeta(x) \pmod{f_i(x)^{k_i}} \quad (i = 1, 2,\ldots, r).$

Since $\zeta(x)$, $\eta_1(x),\ldots, \eta_r(x)$ all satisfy (9), the congruence

$$\sum_{j=1}^{r} \zeta_j(x)^{p^n}\eta_j(x) \equiv \sum_{j=1}^{r} \zeta_j(x)\eta_j(x) \pmod{f(x)}$$

must hold. We consider this congruence, modulo $f_i(x)^{k_i}$. Then (5) and (6) yield the congruence

$$\zeta_i(x)^{p^n} \equiv \zeta_i(x) \pmod{f_i(x)^{k_i}} \quad (i = 1, 2,\ldots, r).$$

This may be written in the form

$$\prod_{\alpha} (\zeta_i(x)-\alpha) \equiv 0 \pmod{f_i(x)^{k_i}} \quad (i = 1, 2,\ldots, r),$$

where the product is taken over all elements α of $\text{GF}(p^n)$. Since each of the polynomial factors $\zeta_i(x)-\alpha$ is relatively prime to all the others, there must be exactly one element α_i of $\text{GF}(p^n)$ for which

$$\zeta_i(x) \equiv \alpha_i \pmod{f_i(x)^{k_i}} \quad (i = 1, 2,\ldots, r).$$

Putting these values of the $\zeta_i(x)$ back into (11) we see that every solution in R of the congruence (9) is of the form (10).

With the aid of (3) and (4) it is quite easy to verify that all expressions of the form (10) form a subring S of R. This completes the proof of Theorem 1.

It is convenient now to regard the ring S as a finite modul over the field $\text{GF}(p^n)$. One basis of this modul is then just the set of polynomials

$$\eta_1(x),\ldots, \eta_r(x).$$

They form an irreducible basis in the sense that a relation of the type

$$\alpha_1 \eta_1(x)+\ldots+\alpha_r \eta_r(x) \equiv 0 \pmod{f(x)}$$

with $\alpha_1, \alpha_2, \ldots, \alpha_r$ in GF(p^n) necessarily implies

$$\alpha_1 = \alpha_2 = \ldots = \alpha_r = 0.$$

For, if we consider this relation, modulo $f_i(x)^{k_i}$, then (5) and (6) lead to

$$\alpha_i \equiv 0 \pmod{f_i(x)^{k_i}},$$

so that

$$\alpha_i = 0$$

since α_i is an element of GF(p^n). Any other irreducible basis of the modul S over GF(p^n) must therefore contain exactly r elements.

4. The determination of a basis of S over GF(p^n)

It will be shown now that the determination of a basis of S over GF(p^n) is a linear problem.

THEOREM 2. *For $j = 1, 2, \ldots, m-1$,*

let

$$x^{jp^n} \equiv \sum_{l=0}^{m-1} A_{jl} x^l \pmod{f(x)}, \tag{12}$$

where the A_{jl} are polynomials in the coefficients of $f(x)$. Let $\delta_{jl} = 0$ if $j \neq l$, and $\delta_{jj} = 1$. Then, the rank s of the matrix

$$\|A_{jl} - \delta_{jl}\| \quad (j = 1, 2, \ldots, m-1; l = 0, 1, \ldots, m-1)$$

is related to the number r by the relation

$$r = m - s.$$

Proof. Let

$$\zeta(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_{m-1} x^{m-1}$$

be any solution in R of (9), with $\alpha_0, \ldots, \alpha_{m-1}$ in GF(p^n). Since $\alpha_i^{p^n} = \alpha_i$ ($i = 0, 1, \ldots, m-1$), we require

$$\sum_{j=0}^{m-1} \alpha_j x^{jp^n} \equiv \sum_{j=0}^{m-1} \alpha_j x^j \pmod{f(x)}.$$

Substituting for x^{jp^n} from (12), we obtain the congruence

$$\alpha_0 - \alpha_0 + \sum_{j=1}^{m-1} \sum_{l=0}^{m-1} \alpha_j x^l (A_{jl} - \delta_{jl}) \equiv 0 \pmod{f(x)}. \tag{13}$$

This shows immediately that α_0 is arbitrary, whilst $\alpha_1, \ldots, \alpha_{m-1}$ are solutions of the m linear, homogeneous equations

$$\sum_{j=1}^{m-1} \alpha_j (A_{jl} - \delta_{jl}) = 0 \quad (l = 0, 1, \ldots, m-1) \tag{14}$$

obtained from (13) by equating to zero the coefficients of each power of x.

With s denoting the rank of the $(m-1)\times m$ matrix $\|A_{jl}-\delta_{jl}\|$ of the system (14), we know that there are exactly $m-s-1$ linearly independent sets of solutions. Together with α_0, there are then exactly $m-s$ sets

$$\alpha_0^{(\rho)}, \alpha_1^{(\rho)},..., \alpha_{m-1}^{(\rho)} \quad (\rho = 1, 2,..., m-s)$$

such that each solution $\zeta(x)$ of (9) is expressible in the form

$$\zeta(x) \equiv \beta_1 \zeta^{(1)}(x)+...+\beta_{m-s}\zeta^{(m-s)}(x) \pmod{f(x)},$$

where $\beta_1,..., \beta_{m-s}$ are in $\mathrm{GF}(p^n)$, and the polynomials

$$\zeta^{(\rho)}(x) = \alpha_0^{(\rho)}+\alpha_1^{(\rho)}x+...+\alpha_{m-1}^{(\rho)}x^{m-1} \quad (\rho = 1, 2,..., m-s)$$

form an irreducible basis of the finite modul S over $\mathrm{GF}(p^n)$. Since any such basis contains exactly r elements, we have proved that $r = m-s$.

5. Application as a reducibility criterion

Theorem 2 relates the number r of distinct irreducible factors of $f(x)$ in $\mathrm{GF}(p^n)$ to the rank s of a certain matrix. If $f(x)$ is a separable polynomial $[k_1 = ... = k_r$ in (1)], r is then the number of irreducible factors of $f(x)$ in $\mathrm{GF}(p^n)$. In this case we can formulate Theorem 2 as a necessary and sufficient reducibility criterion.

THEOREM 3. *Let*

$$f(x) = x^m+\alpha_1 x^{m-1}+...+\alpha_m$$

be a separable polynomial with coefficients in the finite field $\mathrm{GF}(p^n)$. *Then* $f(x)$ *is irreducible over* $\mathrm{GF}(p^n)$ *if and only if the matrix*

$$\|A_{jl}-\delta_{jl}\| \quad (j = 1, 2,..., m-1;\ l = 0, 1,..., m-1)$$

has rank equal to $m-1$; *the elements* A_{jl} *are defined by the congruences*

$$x^{jp^n} \equiv \sum_{l=0}^{m-1} A_{jl}x^l \pmod{f(x)} \quad (j = 1, 2,..., m-1),$$

whilst

$$\delta_{jl} = \begin{cases} 0 & (j \neq l), \\ 1 & (j = l). \end{cases}$$

As an example on this theorem we consider the polynomial

$$f(x) = x^q-\alpha \quad (\alpha \neq 0 \text{ in } \mathrm{GF}(p^n)),$$

where q is a prime number with the congruence property

$$p^n \equiv 1 \pmod{q}.$$

The A_{jl} are very simply calculated in this case: for

$$x^{p^n} = (x^q)^{(p^n-1)/q}x \equiv \alpha^{(p^n-1)/q} x \pmod{f(x)}.$$

Hence $x^{jp^n} \equiv \alpha^{j(p^n-1)/q} x^j \pmod{f(x)}$ $(j = 1, 2, \ldots, q-1)$,

so that $A_{jj} = \alpha^{j(p^n-1)/q}$ $(j = 1, 2, \ldots, q-1)$

and $A_{jl} = 0$ $(j \neq l)$.

The matrix $\|A_{jl} - \delta_{jl}\|$ $(j = 1, 2, \ldots, q-1;\ l = 0, 1, \ldots, q-1)$ has the form

$$\begin{bmatrix} 0 & \alpha^{(p^n-1)/q}-1 & 0 & 0 & \ldots & 0 \\ 0 & 0 & \alpha^{2(p^n-1)/q}-1 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & \ldots & \alpha^{(q-1)(p^n-1)/q}-1 \end{bmatrix}$$

Its rank is $q-1$ if and only if the expression

$$\prod_{\nu=1}^{q-1} \{\alpha^{\nu(p^n-1)/q} - 1\}$$

does not vanish. This requires the non-vanishing of $\alpha^{(p^n-1)/q}-1$: for assume that $\alpha^{\nu(p^n-1)/q} = 1$ for some ν in the range $0 < \nu < q$. Since q is prime, there are integers ρ, σ for which $\rho\nu = q\sigma+1$. Hence

$$\begin{aligned} 1 = \alpha^{\rho\nu(p^n-1)/q} &= \alpha^{(q\sigma+1)(p^n-1)/q} \\ &= \alpha^{\sigma(p^n-1)}\alpha^{(p^n-1)/q} \\ &= \alpha^{(p^n-1)/q} \quad \text{since } \alpha \neq 0 \text{ in GF}(p^n). \end{aligned}$$

Since $x^q - \alpha$ is separable when $\alpha \neq 0$, we have proved that it is irreducible if and only if

$$\alpha^{(p^n-1)/q} \neq 1.$$

For $q = 2$, $n = 1$ this gives Euler's well-known criterion (**2**) for α to be a quadratic non-residue, modulo p.

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TWO MEASURE PROPERTIES OF CARTESIAN PRODUCT SETS

By H. G. EGGLESTON (*Cambridge*)

[Received 12 September 1953]

THE following problem has been suggested by J. Gillis (**2**). *If E is a measurable plane set of Lebesgue measure unity contained in the unit square $S = \{(x, y) \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}$ of a real Euclidean plane, does there exist a non-void perfect set, P, contained in the set $I = \{x \mid 0 \leqslant x \leqslant 1\}$ and a set of positive linear measure Q contained in $G = \{y \mid 0 \leqslant y \leqslant 1\}$ such that $P \times Q \subset E$?*

Here the symbol $(.)\times(.)$ denotes the Cartesian product and the Lebesgue measure of E is plane Lebesgue measure.

In this paper it is shown that such sets P and Q always exist.

An example given by Besicovitch (**1**) shows that the condition of measurability on E cannot be omitted. The result proved here is related to theorems due to S. J. Taylor (**3**).

Gillis [(**2**) 193] constructed an example to show that it may not be possible to find sets P and Q as above for which P as well as Q is of positive linear measure. In § 2, I prove a theorem which is an extension of this example, and show that P, defined above, can be found so as to have positive linear measure provided that $S-E$ is of finite linear measure, and this result is best-possible in that it is not true of all sets E for which $S-E$ is of enumerably infinite linear measure.

1. *Notation.* $\Lambda_i(X)$ denotes the Hausdorff measure of the set X in dimension i. We use this measure only for integral values of i and we suppose that it is normalized so that, if X is a subset of an i-dimensional Euclidean space, then the Hausdorff measure of X coincides with its Lebesgue measure.

If A and B are point sets, $A-B$ is used to denote those points of A that are not in B, whether or not A contains B.

All sets in this section will be subsets of the unit square

$$S = \{(x, y) \mid 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}.$$

If E is any subset of S, $E(x_0)$, defined for $0 \leqslant x_0 \leqslant 1$, denotes the projection upon the y-axis of the intersection of the set E with the line $x = x_0$.

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LEMMA 1. *Given a closed subset of S, say L, a number x_0 $(0 \leqslant x_0 \leqslant 1)$, and a positive number ϵ, then there exists a positive number $\delta = \delta(L; x_0; \epsilon)$, such that*

$$\Lambda_1\Big\{\bigcup_{|x-x_0|\leqslant\delta} L(x)-L(x_0)\Big\} < \epsilon. \tag{1}$$

Since L is closed, we have

$$L(x_0) \supset \bigcap_{\delta>0} \bigcup_{|x-x_0|\leqslant\delta} L(x). \tag{2}$$

It follows that as δ decreases to zero, the set

$$\bigcup_{|x-x_0|\leqslant\delta} L(x)-L(x_0)$$

decreases to the void set. Thus its measure decreases to zero and for δ sufficiently small we have inequality (1).

LEMMA 2. *If E is an F_σ set contained in S and ϵ is a given positive number, then there exists a non-void perfect set P which is a subset of the segment $I = \{x \mid 0 \leqslant x \leqslant 1\}$ and is such that*

$$\Lambda_1\Big\{\bigcup_{x\in P} E(x)\Big\} < \Lambda_2(E)+\epsilon. \tag{3}$$

Suppose that E is the union of the closed sets F_i $(i = 1, 2,...)$. Let K be the subset of points x of I for which

$$\Lambda_1(E(x)) < \alpha+\tfrac{1}{4}\epsilon; \quad \alpha = \Lambda_2(E). \tag{4}$$

K is a set of positive linear measure.

Let $\{i_k\}$ be an increasing sequence of integers so defined that, for all x of a certain subset K_1 of K,

$$\Lambda_1\Big(E(x)-\bigcup_{j=1}^{i_k} F_j(x)\Big) < \epsilon/8^k \quad (k = 1, 2,...), \tag{5}$$

where $\Lambda_1(K_1) > 0$. Write H_k for the set $\bigcup_{j=1}^{i_k} F_j$ and write K_2 for the subset of points of K_1 that have unit metric density with respect to K_1.

Let $x_{1,1}$ be any point of K_2. Since $x_{1,1}$ is a density point of K_1 (and therefore also of K_2), it follows that there is a positive number $\chi(x_{1,1})$ such that, if $0 < \rho < \chi(x_{1,1})$ each of the intervals

$$I_{1,1}(\rho) = \{x \mid x_{1,1}-\tfrac{2}{3}\rho < x < x_{1,1}-\tfrac{1}{3}\rho\},$$
$$J_{1,1}(\rho) = \{x \mid x_{1,1}+\tfrac{1}{3}\rho < x < x_{1,1}+\tfrac{2}{3}\rho\},$$

contains points of K_2.

Write

$$\eta_{1,1} = \min\{\delta(H_1, x_{1,1}, \tfrac{1}{8}\epsilon), \chi(x_{1,1})\}, \tag{6}$$
$$I_{1,1} = I_{1,1}(\eta_{1,1}), \qquad J_{1,1} = J_{1,1}(\eta_{1,1}). \tag{7}$$

Let $x_{1,2}$ be any point of $K_2 \cap I_{1,1}$ and $x_{2,2}$ be any point of $K_2 \cap J_{1,1}$.

Generally, when points $x_{i,j} \in K_2$ $(i = 1, 2, \ldots, 2^{j-1})$ and positive numbers $\eta_{i,j-1}$ $(i = 1, 2, \ldots, 2^{j-2})$ have been defined, we define $x_{i,j+1}$, $\eta_{i,j}$ as follows. Let $\chi(x_{i,j})$ be a positive number such that, for $0 < \rho < \chi(x_{i,j})$, each interval

$$I_{i,j}(\rho) = \{x \mid x_{i,j} - \tfrac{2}{3}\rho < x < x_{i,j} - \tfrac{1}{3}\rho\},$$

$$J_{i,j}(\rho) = \{x \mid x_{i,j} + \tfrac{1}{3}\rho < x < x_{i,j} + \tfrac{2}{3}\rho\},$$

contains points of K_2.

Write
$$\eta_{i,j} = \min\{\delta(H_j, x_{i,j}, \epsilon/8^j), \chi(x_{i,j}), \tfrac{1}{3}\eta_{k,j-1}\}, \tag{8}$$

where k is the integral part of $\frac{1}{2}(1+i)$. Also put

$$I_{i,j} = I_{i,j}(\eta_{i,j}), \qquad J_{i,j} = J_{i,j}(\eta_{i,j}). \tag{9}$$

Select any point of $K_2 \cap I_{i,j}$ and denote it by $x_{2i-1,j+1}$; select any point of $K_2 \cap J_{i,j}$ and denote it by $x_{2i,j+1}$. The sequence of points $x_{i,j}$ and of numbers $\eta_{i,j}$ is thus defined inductively for $j = 1, 2, \ldots$ and $i = 1, 2, \ldots, 2^{j-1}$.

Let P_j denote the union of the closed intervals

$$\{x \mid x_{i,j} - \eta_{i,j} \leqslant x \leqslant x_{i,j} + \eta_{i,j}\} \quad (i = 1, 2, \ldots, 2^{j-1})$$

and let $\bigcap_{j=1}^{\infty} P_j$ be denoted by P. P is a non-void perfect set, and we show that it has the properties required.

Since
$$\bigcup_{x \in P} E(x) = \bigcup_{k=1}^{\infty} \bigcup_{x \in P} H_k(x),$$

we have
$$\Lambda_1\Big(\bigcup_{x \in P} E(x)\Big) = \lim_{k \to \infty} \Lambda_1\Big(\bigcup_{x \in P} H_k(x)\Big) \tag{10}$$

and thus it is sufficient to show that, for every integer k greater than or equal to 4,

$$\Lambda_1\Big(\bigcup_{x \in P} H_k(x)\Big) < \Lambda_2(E) + \tfrac{23}{24}\epsilon. \tag{11}$$

Now $P \subset P_k$ and P_k consists of 2^{k-1} intervals of length $2\eta_{i,k}$ and with centres $x_{i,k}$. By (8), $\eta_{i,k} < \delta(H_k, x_{i,k}, \epsilon/8^k)$ and thus, from Lemma 1,

$$\begin{aligned} \Lambda_1\Big(\bigcup_{x \in P} H_k(x)\Big) &\leqslant \Lambda_1\Big(\bigcup_{x \in P_k} H_k(x)\Big) \\ &\leqslant \Lambda_1\Big\{\bigcup_{i=1}^{2^{k-1}} H_k(x_{i,k})\Big\} + 2^{k-1}\epsilon/8^k. \end{aligned} \tag{12}$$

Also

$$\begin{aligned} \Lambda_1\Big(\bigcup_{i=1}^{2^{k-1}} H_k(x_{i,k})\Big) &\leqslant \Lambda_1\Big(\bigcup_{i=1}^{2^{k-1}} H_{k-1}(x_{i,k})\Big) + \sum_{i=1}^{2^{k-1}} \Lambda_1\Big(H_k(x_{i,k}) - H_{k-1}(x_{i,k})\Big) \\ &\leqslant \Lambda_1\Big(\bigcup_{i=1}^{2^{k-1}} H_{k-1}(x_{i,k})\Big) + 2^{k-1}\epsilon/8^{k-1}, \end{aligned} \tag{13}$$

by (5), since $x_{i,k} \in K_2$ and $H_k \subset E$. Further,

$$\Lambda_1\Big(\bigcup_{i=1}^{2^{k-1}} H_{k-1}(x_{i,k})\Big) < \Lambda_1\Big(\bigcup_{i=1}^{2^{k-2}} H_{k-1}(x_{i,k-1})\Big) + 2^{k-1}\epsilon/8^{k-1}, \tag{14}$$

since both $x_{2i-1,k}$ and $x_{2i,k}$ are at a distance from $x_{i,k-1}$ which is less than $\delta(H_{k-1}, x_{i,k-1}, \epsilon/8^{k-1})$.

From (13), (14) and an inductive argument, we obtain

$$\Lambda_1\Big(\bigcup_{i=1}^{2^{k-1}} H_k(x_{i,k})\Big) < \Lambda_1(H_1(x_{1,1})) + 2\epsilon \sum_{r=2}^{k} 4^{-r+1}. \tag{15}$$

Since $H_1(x_{1,1}) \subset E(x_{1,1})$, (15), (4), and (12) together imply

$$\begin{aligned} \Lambda_1\Big(\bigcup_{x\in P} H_k(x)\Big) &< \alpha + \tfrac{1}{4}\epsilon + \tfrac{2}{3}\epsilon + \epsilon/4^{k-1} \\ &\leqslant \Lambda_2(E) + \tfrac{23}{24}\epsilon. \end{aligned} \tag{16}$$

This completes the proof of the lemma.

The result stated in the introduction can be deduced from Lemma 2 as follows. Let E be a measurable set of planar measure unity, in S. Let D be $S - E$ and G be an open set containing D for which

$$\Lambda_2(G) < \epsilon < \tfrac{1}{4}.$$

G is an F_σ set, and, by Lemma 2, there exists a perfect non-void set P in I such that

$$\Lambda_1\Big(\bigcup_{x\in P} G(x)\Big) < 2\epsilon < \tfrac{1}{2}.$$

But

$$\bigcap_{x\in P} E(x) \supset \bigcap_{x\in P} S(x) - \bigcup_{x\in P} G(x) = Q.$$

Thus $\bigcap_{x\in P} E(x)$ contains a closed subset of positive measure Q. The pair of set P, Q have the properties stated in the introduction.

2. In this section we are concerned with measures in different, but always integral, dimensions.

Let E_n denote n-dimensional real Euclidean space with rectangular coordinates $(x_1, x_2, \ldots, x_n)$. Suppose that r_i are integers

$$1 = r_0 < r_1 < \ldots < r_{k-1} < r_k = n+1,$$

and that E^i is the space $\{(x_1, x_2, \ldots, x_n) \mid x_j = 0 \text{ if } j < r_i \text{ or } j \geqslant r_{i+1}\}$ $(i = 0, 1, \ldots, k-1)$. Let ρ_i be the dimension of E^i and τ be the largest of the ρ_i.

THEOREM. *If X is a subset of*

$$S = \{(x_1, x_2, \ldots, x_n) \mid 0 \leqslant x_j \leqslant 1, j = 1, \ldots, n\}$$

of finite τ-measure, then there exist sets Q_i such that

(i) $Q_i \subset E^i$,

(ii) $\Lambda_{\rho_i}(Q_i) > 0$,

(iii) $Q_0 \times Q_1 \times \ldots \times Q_{k-1} \subset E - X$.

Further, there is a set Y contained in S, of enumerably infinite τ-measure and such that there is no class of sets $Q_0, Q_1, \ldots, Q_{k-1}$ satisfying all the conditions (i), (ii), (iii) *with X replaced by Y.*

To prove the first part of the theorem suppose that the notation is such that E^0 is of dimension τ. The part of the space $E^1 \times E^2 \times \ldots \times E^{k-1}$ contained in S contains an enumerable infinity of non-overlapping non-degenerate intervals $\{I_j\}$. Let A and B be the two projections defined by

$$\left.\begin{aligned} A(x_1, x_2, \ldots, x_n) &= (x_1, x_2, \ldots, x_{r_1-1}, 0, \ldots, 0) \\ B(x_1, x_2, \ldots, x_n) &= (0, 0, \ldots, 0, x_{r_1}, \ldots, x_n) \end{aligned}\right\}. \tag{17}$$

Write

$$K(I_j) = A(X \cap B^{-1}(I_j)). \tag{18}$$

Since

$$\sum_{j=1}^{\infty} \Lambda_\tau(K(I_j)) \leqslant \sum_{j=1}^{\infty} \Lambda_\tau(X \cap B^{-1}(I_j)) \leqslant \Lambda_\tau(X) < \infty, \tag{19}$$

there exists an integer j such that $\Lambda_\tau(K(I_j)) < 1$. The complement of $K(I_j)$ in $E^0 \cap S$ is a set of positive τ-measure, say Q_0. I_j is the product of intervals Q_i, $Q_i \subset E^i$. The k sets $Q_0, Q_1, \ldots, Q_{k-1}$ have the properties (i), (ii), (iii).

To prove the second statement, define τ vectors as follows:

$$\alpha_0 = (\alpha_{1,0}, \alpha_{2,0}, \ldots, \alpha_{n,0}),$$

where $\alpha_{i,0} = 0$ if $i \neq r_j$, and $\alpha_{r_j,0} = 1$ for $j = 0, 1, \ldots, k-1$.

$$\alpha_s = (\alpha_{1,s}, \alpha_{2,s}, \ldots, \alpha_{n,s}),$$

where $\alpha_{i,s} = 0$ unless there exists an integer l, $0 \leqslant l \leqslant k-1$, such that $r_l + s = i < r_{l+1}$, in which case $\alpha_{i,s} = 1$; $s = 1, \ldots, \tau-1$. Let L be the linear space spanned by the vectors $\alpha_0, \alpha_1, \ldots, \alpha_{\tau-1}$ and let $L(\beta)$ be the space obtained from L by a translation of amount equal to the vector β. Let K be the set of all vectors of the n-space all of whose components are rational.

Define Y to be the set $\bigcup_{\beta \in K} L(\beta) \cap S$. Y is of enumerably infinite τ-measure and we shall show that there is no class of sets $Q_0, \ldots, Q_{k-1}$ with the properties (i), (ii), (iii).

Suppose that sets $Q_0,\ldots, Q_{k-1}$ satisfying (i), (ii), (iii) do exist: I shall show that this supposition leads to a contradiction. There exists a point $\gamma = (\gamma_1,\ldots,\gamma_n)$ and intervals K_i of E^i $(i = 0, 1,\ldots, k-1)$ such that the projection of γ on E^i is $P^i(\gamma)$, the centre of K_i, and such that, if R_i is any equi-sided interval of E^i satisfying the conditions

$$K_i \supset R_i \supset P^i(\gamma), \tag{20}$$

then

$$\Lambda_{\rho_i}(Q_i \cap R_i) > (1-\epsilon)\Lambda_{\rho_i}(R_i), \tag{21}$$

where ϵ is a preassigned positive number, which we take to be less than $1/k$.

Let δ be a positive number less than one-half of the least side-length of an interval K_i $(i = 0, 1,\ldots, k-1)$ and also less than one-half of the distance of γ from the frontier of S.

Let λ be a point all of whose coordinates λ_i are rational and which is at a distance from γ of less than $\frac{1}{2}\delta$. Let e_i be the vector whose ith term is unity and all of whose other terms are zero $(i = 1,\ldots, n)$.

The points† $$h = h_0\alpha_0 + \ldots + h_{\tau-1}\alpha_{\tau-1} + \lambda \tag{22}$$

belong to Y for all values of the real numbers h_i satisfying $|h_i| < \frac{1}{2}\delta$ $(i = 0, 1,\ldots, \tau-1)$.

Let h^i be the projection of h on E^i: that is

$$h^i = (h_0+\lambda_{r_i})e_{r_i} + (h_1+\lambda_{r_i+1})e_{r_i+1} + \ldots + (h_{r_{i+1}-r_i-1}+\lambda_{r_{i+1}-1})e_{r_{i+1}-1} \tag{23}$$

$(i = 0, 1,\ldots, k-1)$. Since $h \in Y$, for at least one integer i, $h^i \notin Q_i$.

Let $G^i = T(h^i)$ be the linear subspace of E^0 obtained as follows.

$$G^i = \{(x_1,\ldots,x_n) \mid x_1 = h_0, x_2 = h_1,\ldots, x_{r_{i+1}-r_i} = h_{r_{i+1}-r_i-1};$$
$$x_j = 0 \text{ if } j > \tau; \text{ other } x_j \text{ arbitrary except that } |x_j| < \tfrac{1}{2}\delta\}. \tag{24}$$

Now let the h_j vary subject only to $|h_j| < \frac{1}{2}\delta$. Denote the set described by the point h^i by H^i and write $V^i = T(H^i)$.

Since H^i is an interval of E^i that is equi-sided and satisfies (20), we have from (21)

$$\Lambda_{\rho_i}(Q_i \cap H^i) > (1-\epsilon)\Lambda_{\rho_i}(H^i). \tag{25}$$

Write $Z^i = T(Q_i \cap H^i)$. It can be seen that the k sets V^i $(i = 0,\ldots, k-1)$ are each precisely the set

$$S_\delta = \{(x_1,\ldots,x_n);\ |x_i| < \tfrac{1}{2}\delta,\ i = 1,\ldots,\tau;\ x_i = 0,\ i > \tau\}$$

and that

$$\frac{\Lambda_\tau(Z^i)}{\Lambda_\tau(S_\delta)} = \frac{\Lambda_{\rho_i}(Q_i \cap H^i)}{\Lambda_{\rho_i}(H^i)}. \tag{26}$$

From (25) and (26) applied to $i = 0, 1,\ldots, k-1$,

$$\Lambda_\tau\Big(\bigcap_{i=0}^{k-1} Z^i\Big) > (1-k\epsilon)\Lambda_\tau(S_\delta) > 0. \tag{27}$$

† The terms 'points' and 'vector' are used interchangeably; h is strictly a vector.

But, since for every point h there is an integer i such that $h^i \notin Q_i$, it follows that $\bigcap_{i=0}^{k-1} Z^i$ is void. This is a contradiction of (27).

This completes the proof of the theorem.

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Note added in proof

It has been brought to my notice by Professor A. Zygmund that an affirmative solution of Gillis's problem had been previously obtained by M. L. Brodskiĭ in an article 'On some properties of sets of positive measure' [*Uspehi Matem. Nauk* (N.S.) 4, No. 3 (31), 136–8 (1949)]. The method followed in this article is quite different from that given above and, since Brodskiĭ's own account is confused, a brief version of his proof is given below.

Let G denote the space whose points are measurable linear sets with a distance function defined by $\rho(M,N) = \Lambda_1(M-N)+\Lambda_1(N-M)$, where M and N are any two members of G. We shall suppose that the points of G are subsets of the segment $\{y \mid 0 \leqslant y \leqslant 1\}$. Let $G_{i,n}$ $(i = 1, 2, \ldots)$ be a sequence of open sets covering the whole of G and such that each is of diameter less than $1/n$.

Let E be a closed plane set as described previously in § 1, and let F be a subset of $I\{x \mid 0 \leqslant x \leqslant 1\}$ such that, for some $\epsilon > 0$, $\Lambda_1(E(x)) > \epsilon$ for all $x \in F$. Let $D_{i,n}$ be the subset of F for which $E(x) \in G_{i,n}$ and let $H_{i,n}$ denote the density points of the set $D_{i,n}$, which is a measurable subset of I.

Choose ϵ so small that F is of positive measure. Choose N such that $1/N < \frac{1}{4}\epsilon$. Since $F \subset \bigcup_{i=1}^{\infty} D_{i,N}$, almost all F is contained in $\bigcup_{i=1}^{\infty} H_{i,N}$. Similarly almost all F is contained in $\bigcap_{t=1}^{\infty} \bigcup_{i=1}^{\infty} H_{i,N.t}$. Denote this subset of F by F_1, and its set of density points by F_2.

Choose $x_0 \in F_2$; then $x_0 \in H_{i,N}$ for some i, say i_0. Choose $x_1, x_2 \in F_2 \cap H_{i_0,N}$ so that $x_1 - x_0 = x_0 - x_2 = z_1 > 0$. Next $x_1 \in H_{i,N.4}$ for some i, say i_1; and $x_2 \in H_{i,N.4}$ for some i, say i_2. Choose $x_{11}, x_{12} \in H_{i_1,N.4} \cap F_2$ and $x_{21}, x_{22} \in H_{i_2,N.4} \cap F_2$ and

$$x_{11} - x_1 = x_1 - x_{12} = x_{21} - x_2 = x_2 - x_{22} = z_2 > 0, \quad z_2 < \tfrac{1}{3}z_1.$$

The definition of the points $x_{j_1,j_2,\ldots,j_r} \in H_{i,N.4^r}$ $(j_k = 1 \text{ or } 2)$ is continued inductively. From the definition of the points $x_{j_1,\ldots,j_r}$,

$$\Lambda_1(E(x_{j_1,\ldots,j_r,j_{r+1}}) \cap E(x_{j_1,\ldots,j_r})) \geqslant \Lambda_1(E(x_{j_1,\ldots,j_r})) - \Lambda_1(E(x_{j_1,\ldots,j_r}) - E(x_{j_1,\ldots,j_r,j_{r+1}}))$$

$$\geqslant \Lambda_1(E(x_{j_1,\ldots,j_r})) - 4^{-r}\epsilon.$$

It follows from successive applications of this and similar inequalities that, if T denotes the intersection of all the sets $E(x_0)$, $E(x_1)$, $E(x_2)$,..., $E(x_{j_1,\ldots,j_r})$,..., then

$$\Lambda_1(T) \geqslant \Lambda_1(E(x_0)) - 2(4^{-1}\epsilon) - 4(4^{-2}\epsilon) - 8(4^{-3}\epsilon) - \ldots > 0.$$

Further, since E is closed, any limit point x of the set of points $x_0, x_1, x_2, \ldots, x_{j_1,\ldots,j_r}$, is such that $E(x) \supset T$. The set of these points is a perfect set, say P, and $T \times P \subset E$. Thus the required result is proved.

I am indebted to Professor A. S. Besicovitch for translating this paper for me.

RECIPROCAL STATIC SOLUTIONS OF THE EQUATIONS $G_{\mu\nu} = 0$

By H. A. BUCHDAHL (*Hobart*)

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1. In the General Theory of relativity empty space-time is taken to be a four-dimensional Riemann space V_4 whose Ricci tensor $G_{\mu\nu}$ vanishes. Further, a solution

$$ds^2 = g_{\mu\nu}\,dx^\mu dx^\nu \tag{1.1}$$

of the equations

$$G_{\mu\nu} = 0 \tag{1.2}$$

is called *static* if the $g_{\mu\nu}$ satisfy the conditions g_{14}, g_{24}, $g_{34} = 0$ and $g_{\mu\nu,4} = 0$ ($\mu, \nu = 1,\ldots, 4$); x^4 is the coordinate time, and subscripts after a comma indicate ordinary differentiation. Generalizing to a V_n, I call the line element (1.1) *static* if there is a coordinate x^a such that the $g_{\mu\nu}$ satisfy the conditions

$$g_{ia} = 0, \qquad g_{\mu\nu,a} = 0, \tag{1.3}$$

i.e. if (1.1) has the form

$$ds^2 = g_{ik}(x^j)\,dx^i dx^k + g_{aa}(x^j)\,(dx^a)^2. \tag{1.4}$$

Greek indices take the values $1,\ldots, n$, whereas roman indices run over the same range but with a omitted. (The index a being fixed no summation over it is to be carried out even when it is repeated.) I now prove in this paper the following theorem:

THEOREM. *If* (1.4) *is a given static solution of the equations* (1.2) *for a* V_n ($n \geqslant 4$), *then another static solution is immediately given by*

$$ds^2 = (g_{aa})^{2/(n-3)} g_{ik}\,dx^i dx^k + (g_{aa})^{-1}\,(dx^a)^2. \tag{1.5}$$

Any two solutions related in this way will be called *reciprocal* to one another.

2. Let

$$ds^2 = \bar{g}_{ik}\,dx^i dx^k + e^{2\gamma}(dx^a)^2 \tag{2.1}$$

be a static line-element. Then it follows that the Ricci tensor associated with the V_n can be written (1)

$$\left.\begin{aligned} G_{ik} &= \bar{G}_{ik} + \beta_{ik} \\ G_{ia} &= 0 \\ G^a_a &= \bar{g}^{ik}\beta_{ik} = \beta \end{aligned}\right\}, \tag{2.2}$$

where

$$\beta_{ik} = \gamma_{:ik} + \gamma_{:i}\gamma_{:k}. \tag{2.21}$$

Here $\bar{G}_{ik}$ is the Ricci tensor of the V_{n-1} whose metric is $ds^2=\bar{g}_{ik}\,dx^i dx^k$, and subscripts after a colon denote covariant differentiation in this V_{n-1}. Now write $\bar{g}_{ik}=e^{2\sigma}\underset{0}{g}_{ik}$ and let subscripts following a semicolon denote covariant differentiation in the $\underset{0}{V}_{n-1}$ whose metric is $ds^2=\underset{0}{g}_{ik}\,dx^i dx^k$. Since V_{n-1} is conformal to $\underset{0}{V}_{n-1}$ it follows (**2**) that

$$\bar{G}_{ik}=\underset{0}{G}_{ik}+(n-3)(\sigma_{;ik}-\sigma_{;i}\,\sigma_{;k})+[\Delta\sigma+(n-3)\widehat{\sigma\sigma}]. \tag{2.3}$$

For any two scalars ξ and η, $\widehat{\xi\eta}$ denotes the quantity $\underset{0}{g}^{ik}\xi_{;i}\,\eta_{;k}$, whilst $\Delta\xi$ stands for $\underset{0}{g}^{ik}\xi_{;ik}$, and $\underset{0}{G}_{ik}$ is the Ricci tensor of $\underset{0}{V}_{n-1}$. Moreover

$$\left.\begin{aligned}\gamma_{:i}&=\gamma_{,i}=\gamma_{;i}\\ \gamma_{:ik}&=\gamma_{,ik}-\overline{\Gamma}^{j}_{ik}\gamma_{,j}=\gamma_{;ik}-(\sigma_{;i}\,\gamma_{;k}+\sigma_{;k}\,\gamma_{;i})+\underset{0}{g}_{ik}\widehat{\sigma\gamma}\end{aligned}\right\}. \tag{2.4}$$

When we put (2.3), (2.4) in (2.1), the Ricci tensor of V_n becomes

$$\left.\begin{aligned}G_{ik}&=\underset{0}{G}_{ik}+(n-3)(\sigma_{;ik}-\sigma_{;i}\,\sigma_{;k})+\gamma_{;ik}+\gamma_{;i}\,\gamma_{;k}-\\ &\qquad-(\sigma_{;i}\,\gamma_{;k}+\sigma_{;k}\,\gamma_{;i})+\underset{0}{g}_{ik}[\Delta\sigma+(n-3)\widehat{\sigma\sigma}+\widehat{\sigma\gamma}]\\ G_{ia}&=0\\ G^{a}_{a}&=e^{-2\sigma}[\Delta\gamma+\widehat{\gamma\gamma}+(n-3)\widehat{\sigma\gamma}]\end{aligned}\right\}. \tag{2.5}$$

From (2.5) there follow the equations

$$\left.\begin{aligned}e^{2\sigma}\underset{0}{g}^{ik}G_{ik}&=\underset{0}{G}+2(n-2)\Delta\sigma+\Delta\gamma+(n-2)(n-3)\widehat{\sigma\sigma}+(n-3)\widehat{\sigma\gamma}+\widehat{\gamma\gamma}\\ e^{2\sigma}G^{a}_{a}&=\qquad\qquad\Delta\gamma\qquad\qquad+(n-3)\widehat{\sigma\gamma}+\widehat{\gamma\gamma}\end{aligned}\right\}, \tag{2.6}$$

where $\underset{0}{G}$ is the scalar curvature $\underset{0}{g}^{ik}\underset{0}{G}_{ik}$ of V_{n-1}.

3. Suppose now that $G_{\mu\nu}=0$, so that the left-hand members of the equations (2.6) vanish. Then (i) add the second equation of (2.6) to the first after multiplying the former throughout by $(3n-5)/(n-3)$; alternatively (ii) subtract the second equation of (2.6) from the first. One then arrives at the equations

$$\left.\begin{aligned}\Delta\mu+\widehat{\mu\mu}+\frac{n-3}{4(n-2)}\underset{0}{G}&=0\\ \Delta\lambda+\widehat{\lambda\lambda}+\frac{n-3}{4(n-2)}\underset{0}{G}&=0\end{aligned}\right\}, \tag{3.1}$$

where
$$\lambda=\tfrac{1}{2}(n-3)\sigma,\qquad \mu=\tfrac{1}{2}(n-3)\sigma+\gamma. \tag{3.2}$$

In terms of λ and μ the first member of (2.5) now yields

$$G_{ik} = 0 = \underset{0}{G_{ik}} + \lambda_{;ik} + \mu_{;ik} + \lambda_{;i}\lambda_{;k} - \frac{n-1}{n-3}(\lambda_{;i}\mu_{;k} + \lambda_{;k}\mu_{;i}) + \\ + \mu_{;i}\mu_{;k} + 2\underset{0}{g_{ik}}\left(\frac{1}{n-3}\widehat{\lambda\mu} - \frac{1}{4(n-2)}\underset{0}{G}\right), \quad (3.3)$$

where the coefficient of $\underset{0}{g_{ik}}$ has been simplified by means of (3.1). Inspection of (3.1), (3.3) shows that the equations $G_{\mu\nu} = 0$ are invariant with respect to the mutual interchange of λ and μ.

Suppose now that (1.4) is a given static solution of (1.2): in other words that (3.1), (3.3) are satisfied by $\sigma = 0$, $\gamma = \frac{1}{2}\log g_{aa}$,

i.e. $$\lambda = 0, \qquad \mu = \tfrac{1}{2}\log g_{aa}.$$

Then it follows from the result above that the same equations are satisfied by $\lambda = \frac{1}{2}\log g_{aa}$, $\mu = 0$,

i.e. $$\sigma = \frac{1}{n-3}\log g_{aa}, \qquad \gamma = -\tfrac{1}{2}\log g_{aa},$$

so that (1.5) is also a solution of (1.4), which was to be proved.

4. It is convenient to refer to the process of formation of a reciprocal solution as a *reciprocal transformation*. Then it is obvious that in certain cases a solution reciprocal to a given solution will also be obtainable from the latter by means of a transformation of coordinates. This is the case, for example, if one applies to the general spherically symmetric solution of (1.2) a reciprocal transformation which does not destroy spherical symmetry. Consider in particular the Schwarzschild line element ($n = 4$), viz. (3)

$$ds^2 = -(\gamma^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2) + \gamma\, dt^2, \quad (4.1)$$

where $\gamma = 1 - 2m/r$, m = constant. The solution reciprocal to (4.1) is then

$$ds^2 = -\gamma^2(\gamma^{-1}dr^2 + r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2) + \gamma^{-1}dt^2, \quad (4.2)$$

x^a having been chosen to be x^4 $(= t)$. The coordinate transformation $r = -\bar{r} + 2m$ then reproduces (4.1), if we write $\bar{r}$ in place of r in the latter.

As an elementary example of a non-trivial reciprocal transformation consider the line element

$$ds^2 = -(dx^2 + dy^2 + dz^2) + x^2 dt^2, \quad (4.3)$$

which is obviously that of a flat space, and therefore a solution of (1.2). The reciprocal solution

$$ds^2 = -x^4(dx^2+dy^2+dz^2)+x^{-2}dt^2 \tag{4.4}$$

is, however, the metric of a V_4, i.e. not an E_4 [cf. (4)]. More general solutions of (1.2) can be formed by means of a succession of reciprocal transformations, starting with the line element of a flat space.

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THE FINITE STURM–LIOUVILLE TRANSFORM

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1. Introduction

It is known that certain partial differential equations can be solved by use of particular types of definite integrals having appropriate kernels. The choice of kernel depends on the type of boundary-value problem. The solution obtained by these transforms is direct in the sense that it contains the boundary values in the solution. This, of course, is lacking in the classical approach. Finite Fourier transforms are of this type (**1**).

Recently Tranter (**2**) used a Legendre polynomial as a kernel. Scott (**3**) following a similar approach used a Jacobi polynomial as a kernel which extends the result of (**2**).

It is the purpose of the present paper to extend and unify all such special transforms. Thus we employ a kernel which may be determined to suit each particular type of problem. The transform can be employed to solve a wide class of linear second-order partial differential equations.

In this paper I deal only with those transforms whose intervals are finite. Thus, the results obtained here are particularly useful for finite domains, or domains which are finite in one direction. An extension to infinite domains and singular cases will be made in a later paper.

2. The Sturm–Liouville expansion

A second-order, linear, homogeneous differential equation containing an arbitrary parameter λ has the general form

$$Mv \equiv p_0 v_{yy} + p_1 v_y + (p_2 + \lambda p_3)v = 0 \quad (a_0 \leqslant y \leqslant b_0), \tag{1}$$

where $p_i = p_i(y)$; v and y are the dependent and real independent variables, respectively, and $\phi_y = d\phi/dy$, etc.

Equation (1) can be transformed into the canonical form:

$$Lu = \lambda u, \qquad L \equiv q(x) - \frac{d^2}{dx^2} \quad (a \leqslant x \leqslant b), \tag{2}$$

where

$$\left.\begin{aligned} u &= \theta(x)v, & x &= \int (p_3/p_0)^{\frac{1}{2}}\, dy \\ a &= \int^{a_0} (p_3/p_0)^{\frac{1}{2}}\, dy, & b &= \int^{b_0} (p_3/p_0)^{\frac{1}{2}}\, dy \\ q(x) &= \frac{\theta_{xx}(x)}{\theta(x)} - \phi(x), & \theta(x) &= (p_3 k^2/p_0)^{\frac{1}{2}} \\ \phi(x) &= p_2/p_3, & k &= \exp \int (p_1/p_0)\, dy \end{aligned}\right\}. \tag{3}$$

It is simpler to work with (2).

Quart. J. Math. Oxford (2), 5 (1954), 120–9.

We consider two solutions $\phi(x,\lambda)$ and $\chi(x,\lambda)$ of (2) satisfying the boundary conditions

$$\left.\begin{aligned} \phi(a,\lambda) &= \sin\alpha, \qquad \phi_x(a,\lambda) = -\cos\alpha \\ \chi(b,\lambda) &= \sin\beta, \qquad \chi_x(b,\lambda) = -\cos\beta \end{aligned}\right\}. \tag{4}$$

Both solutions are unique if $q(x)$ is real, is continuous everywhere in (a,b), and has finite limits for $x = a$ and $x = b$ [see (4) 6]. We can build up $\phi(x,\lambda)$ and $\chi(x,\lambda)$ as a linear combination of two independent solutions $\phi_0(x,\lambda)$ and $\chi_0(x,\lambda)$ of (2). For let

$$\left.\begin{aligned} \omega_0(\lambda)\phi(x,\lambda) &= \phi_0(x,\lambda)[\chi_0(a,\lambda)\cos\alpha+\chi_{0,x}(a,\lambda)\sin\alpha]- \\ &\quad -\chi_0(x,\lambda)[\phi_0(a,\lambda)\cos\alpha+\phi_{0,x}(a,\lambda)\sin\alpha], \\ \omega_0(\lambda)\chi(x,\lambda) &= \phi_0(x,\lambda)[\chi_0(b,\lambda)\cos\beta+\chi_{0,x}(b,\lambda)\sin\beta] \\ &\quad -\chi_0(x,\lambda)[\phi_0(b,\lambda)\cos\beta+\phi_{0,x}(b,\lambda)\sin\beta], \\ \omega_0(\lambda) &= W(\phi_0,\chi_0) = \phi_0(x,\lambda)\chi_{0,x}(x,\lambda)-\phi_{0,x}(x,\lambda)\chi_0(x,\lambda) \end{aligned}\right\}. \tag{5}$$

Functions $\phi(x,\lambda)$ and $\chi(x,\lambda)$ defined by (5) satisfy (2) and (4). Here $W(u,v)$ is the Wronskian and *can be shown to be independent of x.*

Let λ_n be a root of $W(\phi,\chi) = 0$. Then we have

$$\omega(\lambda_n) = W(\phi_n,\chi_n) \equiv \phi_n\chi_{n,x}-\phi_{n,x}\chi_n = 0,$$

i.e.
$$\phi_{n,x}/\phi_n = \chi_{n,x}/\chi_n,$$

where
$$\phi_n \equiv \phi(x,\lambda_n) \quad \text{and} \quad \chi_n \equiv \chi(x,\lambda_n).$$

Hence the integration gives

$$\chi(x,\lambda_n) = k_n\phi(x,\lambda_n). \tag{6}$$

Consequently $\phi(x,\lambda_n)$ and $k_n\phi(x,\lambda_n)$ *satisfy all of the boundary conditions* (4).

It is also easy to show that $\phi(x,\lambda_n)$ *is an orthogonal set.* For, in view of (2), (4), and (6), we have

$$(\lambda_m-\lambda_n)\int_a^b \phi_m\phi_n\,dx$$

$$= \int_a^b (\phi_m L\phi_n-\phi_n L\phi_m)\,dx = W(\phi_m,\phi_n) = 0 \quad (\lambda_m \neq \lambda_n).$$

Hence we can write

$$\int_a^b \phi(x,\lambda_n)\phi(x,\lambda_m)\,dx = N^2\delta_{mn}, \tag{7}$$

where δ_{mn} is the Kronecker delta (equal to zero when $m \neq n$, to unity when $m = n$).

It can be shown also that *all roots* of $\omega(\lambda) = 0$ *are real and distinct* [see (**4**) 11–12].

From (5) and (6) it follows that

$$k_n = \frac{\phi_0(b,\lambda_n)\cos\beta + \phi_{0,x}(b,\lambda_n)\sin\beta}{\phi_0(a,\lambda_n)\cos\alpha + \phi_{0,x}(a,\lambda_n)\sin\alpha} = \frac{\chi_0(b,\lambda_n)\cos\beta + \chi_{0,x}(b,\lambda_n)\sin\beta}{\chi_0(a,\lambda_n)\cos\alpha + \chi_{0,x}(a,\lambda_n)\sin\alpha}. \tag{8}$$

If now $f(x)$ *is an integrable function over* (a,b) *and if* $a < x < b$, *we have the Sturm–Liouville expansion of* $f(x)$

$$f(x) = \sum_{n=0}^{\infty} [k_n/\omega_\lambda(\lambda_n)]\phi(x,\lambda_n) \int_a^b \phi(y,\lambda_n) f(y)\, dy. \tag{9}$$

[For the convergence of (9) to $f(x)$ see (**4**) 12–15 or (**5**) 275–6.]

3. Finite Sturm–Liouville transforms

Let $f(x)$ *be a real continuous and integrable function of* x, *in the interval* (a,b). *We define the finite Sturm–Liouville transform* $S\{f\} \equiv \bar{f}(\lambda_n)$ *associated with system* (2) *and* (4) *by*

$$\bar{f}(\lambda_n) \equiv S\{f\} = \int_a^b f(y)\psi(y,\lambda_n)\, dy, \tag{10}$$

where

$$\psi(y,\lambda_n) = [k_n/\omega_\lambda(\lambda_n)]^{\frac{1}{2}}\phi(y,\lambda_n). \tag{11}$$

The inversion theorem now follows from (9), namely

$$f(x) = \sum_{n=0}^{\infty} \bar{f}(\lambda_n)\psi(x,\lambda_n). \tag{12}$$

Here functions $\psi(y,\lambda_n)$ are orthonormal,

i.e.

$$\int_a^b \psi(x,\lambda_n)\psi(x,\lambda_m)\, dx = \delta_{mn}. \tag{13}$$

This is readily seen from (7) and (11). Equations (10) and (12) are basic for the solution of some partial differential equations.

4. Transform of Lf

We are now going to prove that

$$S_x\{Lf\} \equiv S_x\{q(x)f - \partial^2 f/\partial x^2\} = B_f(\lambda_n,t) + \lambda_n \bar{f}(\lambda_n,t), \tag{14}$$

where

$$B_f(\lambda_n,t) = [k_n/\omega_\lambda(\lambda_n)]^{\frac{1}{2}} \times \times \left\{ -f(b,t)\frac{\cos\beta}{k_n} - f_x(b,t)\frac{\sin\beta}{k_n} + f(a,t)\cos\alpha + f_x(a,t)\sin\alpha \right\} \tag{15}$$

is a function depending on the boundary values of $f(x,t)$, $f_x(x,t)$ at $x=a,b$. Subscript f implies that the function B contains boundary values of the function f. Here S_x represents the finite Sturm–Liouville transform taken with respect to x,

$$S_x\{\phi(x,t)\} = \bar{\phi}(\lambda_n,t). \tag{16}$$

To prove (14) we multiply Lf by $\psi(x,\lambda_n)$ and integrate between (a,b): that is, by definition we have

$$S_x\{Lf\} \equiv \int_a^b [q(x)f(x,t) - \partial^2 f/\partial x^2]\psi(x,\lambda_n)\,dx. \tag{17}$$

Integrating the second term in the integrand of (17) twice by parts we obtain

$$S_x\{Lf\} = -[f_x\psi - \psi_x f]_a^b + \int_a^b f(x,t)L\psi(x,\lambda_n)\,dx. \tag{18}$$

If we now use (4) and (6) in the first term of the right-hand side and (2) inside the integrand, we obtain (14).

5. Solution of some partial differential equations

(i) Let us consider a *parabolic* partial differential equation

$$Mv + p_4 v_t = 0, \tag{19}$$

where Mv is defined by (1), $p_i = p_i(y,t)$ are known, and $v(y,t)$ is the unknown dependent variable.

The use of transformation (3) reduces (19) to

$$u_{xx} + [\lambda - p(x,t)]u + (p_4/p_3)u_t = 0, \tag{20}$$

where

$$p(x,t) = q(x,t) - p_4 p_0 (k/2p_0)^2 \frac{\partial}{\partial t}(p_0/p_3 k^2). \tag{21}$$

Here $q(x,t)$, $k(t)$, and $\lambda(t)$ are given by (3) with t as a parameter. If $\phi(x,t,\lambda_n)$ and $\psi(x,t,\lambda_n)$ are now obtained as before except with $q(x)$ replaced by $p(x,t)$, where t is a parameter and p_4/p_3 a function of t alone, then (20), with the use of the finite Sturm–Liouville transform, can be reduced to an ordinary differential equation in t. For, let $p_4/p_3 = r(t)$; then application of the transform to (20) gives

$$-B_u(\lambda_n,t) + (\lambda - \lambda_n)\bar{u}(\lambda_n,t) + r(t)\bar{u}_t(\lambda_n,t) = 0. \tag{22}$$

Equation (22) is of first order. The complete solution is

$$\bar{u}(\lambda_n,t) = C(\lambda_n)\exp\left(\int^t \frac{\lambda_n - \lambda}{r}\,dt\right) + \\ + \left[\exp\left(\int^t \frac{\lambda_n - \lambda}{r}\,dt\right)\right]\int^t (B_u/r)\left[\exp\left(-\int^t \frac{\lambda_n - \lambda}{r}\,dt\right)\right]dt. \tag{23}$$

Now, the inversion theorem (12) gives

$$u(x,t) = \sum_{n=0}^{\infty} \bar{u}(\lambda_n, t)\psi_n(x,t,\lambda_n). \tag{24}$$

Therefore the solution of (19) is effected. We must, however, remember that at least one of each pair of boundary values $u(a,t)$, $u_x(a,t)$ and $u(b,t)$, $u_x(b,t)$ must be known. By selecting α and β properly in $B_u(\lambda_n,t)$, we can make the terms containing the unknown pair zero. Of course any knowledge of the boundary values of u leading to the complete evaluation of B_u is sufficient for the solution.

(ii) A second-order linear partial differential equation containing a parameter λ has the general form

$$\left.\begin{aligned} Nv+a_4 v_z+a_5 v_\tau+(a_6+\lambda a_7)v = 0 \\ Nv = a_1 v_{zz}+2a_2 v_{z\tau}+a_3 v_{\tau\tau} \end{aligned}\right\}, \tag{25}$$

where $a_i = a_i(z,\tau)$. If the equation is of *elliptic type*, we have

$$a_1 a_3 - a_2^2 > 0. \tag{26}$$

Consider now an elliptic equation (25). Let $\xi(z,\tau) =$ constant and $\eta(z,\tau) =$ constant, respectively, be the solutions of the differential equations

$$d\tau/dz = \zeta_1(z,\tau), \qquad d\tau/dz = \zeta_2(z,\tau),$$

$$\zeta_1, \zeta_2 = -\frac{a_2}{a_1} \pm i\left[\frac{a_3}{a_1} - \left(\frac{a_2}{a_1}\right)^2\right]^{\frac{1}{2}}. \tag{27}$$

If we set

$$2y = \xi+\eta, \qquad 2it = \xi-\eta, \tag{28}$$

equation (25) can be transformed into the canonical form (6)

$$v_{yy}+p_1 v_y+(p_2+\lambda p_3)v+p_4 v_t+v_{tt} = 0, \tag{29}$$

where

$$p_1 = \frac{2}{\beta}Py, \qquad p_2 = 2a_6/\beta, \qquad p_3 = 2a_7/\beta,$$

$$p_4 = \frac{2}{\beta}Pt, \qquad \beta = a_1(y_z^2+t_z^2)+2a_2(y_z y_\tau+t_z t_\tau)+a_3(y_\tau^2+t_\tau^2). \tag{30}$$

Here the operator P is defined by

$$P\phi = N\phi+a_4\phi_z+a_5\phi_\tau \tag{31}$$

and $N\phi$ is given by the second equation of (25).

On using the transformation (3) with $p_0 = 1$, (29) becomes

$$u_{xx}+(\lambda-p)u+r_1 u_t+r_0 u_{tt} = 0, \tag{32}$$

where

$$\left.\begin{aligned} p = q(x,t)-mp_4 g_t-mg_{tt}, \qquad r_0 = p_3^{-1} \\ r_1 = mp_4 g+2mg_t, \qquad g = (p_3 k^2)^{-1} \\ m = (k^2/p_3^3)^{\frac{1}{4}} \end{aligned}\right\}. \tag{33}$$

In the special case where r_1 and r_0 are functions of t alone, equation (32) can be transformed into an ordinary differential equation by applying the finite Sturm–Liouville transform. Hence

$$r_0\,\bar{u}_{tt}+r_1\,\bar{u}_t+(\lambda-\lambda_n)\bar{u}(\lambda_n,t) = B_n(\lambda_n,t). \tag{34}$$

We can now either use transformation (3) with respect to t, with r_0, r_1 replacing p_0, p_1, $p_2 = 0$, $p_3 = 1$ and find the solution of (34) and then invert it, or solve (34) directly for $\bar{u}(\lambda_n, t)$, and then (24) gives $u(x,t)$.

It must be remembered that this method has the advantage that it contains the boundary values of the function within the solution; consequently, the difficulty of satisfying boundary conditions is eliminated. The method is also valid in the cases where the usual technique of *separation of variables* fails.

6. Solutions ϕ_0 and χ_0 of some special equations

Below are given two independent solutions ϕ_0 and χ_0 of some special differential equations (D.E.) of mathematical physics. Their canonical forms (2) will be given by determining only u, q, and λ.

(i) *Bessel Functions.*

$$v_{yy}+\frac{1}{y}v_y+\left(s^2-\frac{\nu^2}{y^2}\right)v = 0, \qquad \text{(D.E.)}$$

$$u = x^{\frac{1}{2}}v, \qquad y = x, \qquad q = (\nu^2-\tfrac{1}{4})x^{-2}, \qquad \lambda = s^2,$$

$$\phi_0(x,\lambda) = x^{\frac{1}{2}}J_\nu(xs), \qquad \chi_0(x,\lambda) = x^{\frac{1}{2}}Y_\nu(xs).$$

Interval (a,b) should not contain $x = 0$.

(ii) *Spherical Harmonics.*

$$[(1-y^2)v_y]_y+\left[\nu(\nu+1)-\frac{\mu^2}{1-y^2}\right]v = 0, \qquad \text{(D.E.)}$$

$$u = (\cos x)^{\frac{1}{2}}v, \qquad y = \sin x, \qquad q = \mu^2\sec^2 x-\tfrac{1}{4}\tan^2 x-\tfrac{1}{2},$$

$$\lambda = \nu(\nu+1), \qquad \phi_0(x,\lambda) = (\cos x)^{\frac{1}{2}}\,P_\nu^\mu(\sin x),$$

$$\chi_0(x,\lambda) = (\cos x)^{\frac{1}{2}}\,Q_\nu^\mu(\sin x).$$

Interval (a,b) should not contain $(-1,1)$.

(iii) *Hermite Polynomials.*

$$v_{yy}-yv_y+nv = 0, \qquad \text{(D.E.)}$$

$$u = e^{-\frac{1}{4}x^2}v, \qquad y = x, \qquad q = -\tfrac{1}{2}+\tfrac{1}{4}x^2, \qquad \lambda = n,$$

$$\phi_0(x,\lambda) = e^{-\frac{1}{4}x^2}\mathrm{He}_n(x), \qquad \chi_0(x,\lambda) = e^{-\frac{1}{4}x^2}\mathrm{he}_n(x).$$

(iv) *Tschebyscheff Polynomials.*

$$(1-y^2)v_{yy}-yv_y+n^2v=0, \qquad \text{(D.E.)}$$

$$u=v, \qquad y=\sin x, \qquad q=0, \qquad \lambda=n^2,$$

$$\phi_0(x,\lambda)=T_n(\sin x)=\cos n(\sin^{-1}y),$$

$$\chi_0(x,\lambda)=U_n(\sin x)=\sin n(\sin^{-1}y).$$

(v) *Mathieu Functions.*

$$u_{xx}+(\lambda-2h^2\cos 2x)u=0, \qquad \text{(D.E.)}$$

$$\phi_0(x,\lambda)=\mathrm{ce}_{2n}(x), \qquad \mathrm{ce}_{2n+1}(x), \qquad \mathrm{se}_{2n}(x), \qquad s_{2n+1}(x),$$

$$\chi_0(x,\lambda)=\mathrm{ce}^{(2)}_{2n}(x), \qquad \mathrm{ce}^{(2)}_{2n+1}(x), \qquad \mathrm{se}^{(2)}_{2n}(x), \qquad s^{(2)}_{2n+1}(x)$$

($n=0, 1, 2,\ldots$, but for se_{2n} only $n=1, 2,\ldots$). $\phi_0(x,\lambda)$ is periodic.

(vi) *Whittaker Functions.*

$$v_{yy}+\{-\tfrac{1}{4}+ky^{-1}+(\tfrac{1}{4}-\mu^2)y^{-2}\}v=0, \qquad \text{(D.E.)}$$

$$u=(\tfrac{1}{2}x)^{\frac{1}{2}}v, \qquad y=\tfrac{1}{4}x^2, \qquad q=\tfrac{1}{4}(16\mu^2-17)x^{-2}+\tfrac{1}{16}x^2,$$

$$\phi_0(x,\lambda)=(\tfrac{1}{2}x)^{\frac{1}{2}}W_{k,\mu}(\tfrac{1}{4}x^2) \qquad \chi_0(x,\lambda)=(\tfrac{1}{2}x)^{\frac{1}{2}} \qquad W_{-k,\mu}-(\tfrac{1}{4}x^2).$$

The interval (a,b) should not contain $x=0$.

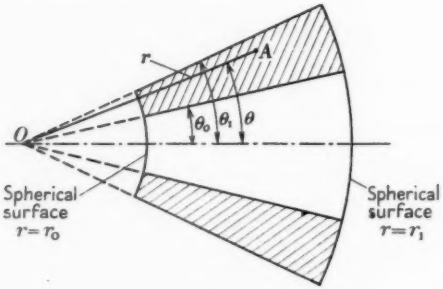

FIG. 1.

7. Application

As an illustration we solve a heat-conduction problem which may be looked upon as a mathematical model of *volcanoes*. The problem also has application in the exhaust ports of jet engines. As far as I know the problem has not previously been solved.

Consider the conical shell enclosed by two coaxial cones having the same apex O, and two concentric spheres having O as their centre (Fig. 1).

In polar coordinates the inner and outer conical surfaces are given by $\theta=\theta_0$ and $\theta=\theta_1$ and the end surfaces by $r=r_0$ and $r=r_1$.

PROBLEM. *To determine the steady temperature-distribution within the shell under the general axi-symmetric boundary conditions in temperature.*

This is the general Dirichlet problem for the domain under consideration. The differential equation and boundary conditions (B.C.) are given below:

$$\text{(D.E.)}\quad \Delta V = r^{-2}(r^2V_r)_r + r^{-2}[(1-y^2)V_y]_y = 0 \quad (y=\cos\theta;\ r_0<r<r_1;\ \theta_0<\theta<\theta_1). \tag{35}$$

$$\text{(B.C.)}\quad \left.\begin{aligned} V &= V_0(r) \quad (\theta=\theta_0)\\ V &= V_1(r) \quad (\theta=\theta_1)\end{aligned}\right\} \quad (r_0<r<r_1), \tag{36}$$

$$\left.\begin{aligned} V &= V_3(\theta) \quad (r=r_0)\\ V &= V_4(\theta) \quad (r=r_1)\end{aligned}\right\} \quad (\theta_0<\theta<\theta_1).$$

Here $V(r,y)$ is the temperature function, r and θ are the polar coordinates.

Solution. We can exclude the second term of (D.E.) (35) if we use a finite Sturm–Liouville transform associated with the Legendre equation

$$[(1-y^2)v_y]_y+\nu(\nu+1)v = 0. \tag{37}$$

In view of § 6 (ii) with $\mu = 0$ we find that, if we select

$$y = \sin x, \qquad u = (\cos x)^{\frac{1}{2}}v,$$

we can transform (37) to canonical form leading to solutions

$$\phi_0(x,\lambda) = (\cos x)^{\frac{1}{2}}P_\nu(\sin x), \quad \chi_0(x,\lambda) = (\cos x)^{\frac{1}{2}}Q_\nu(\sin x), \quad \lambda = \nu(\nu+1), \tag{38}$$

where P_ν and Q_ν are Legendre functions of the first and second kind respectively. In view of (14), when we apply the transform to (35), the second term gives $-\nu_n(\nu_n+1)-B_v(r,\lambda_n)$. Now B_v contains four arbitrary functions which must be specified on the surfaces $\theta=\theta_0$ and $\theta=\theta_1$. The terms containing the derivatives of v are not given. Thus, if we select $\alpha=\beta=0$, these terms drop out, leaving the terms containing $v_0(r)$ and $v_1(r)$, which are given. Hence $\phi(x,\lambda)$ and $\omega_0(\lambda)$ of (5) become

$$\phi(x,\lambda) = (\cos x_0\cos x)^{\frac{1}{2}}[P_\nu(\sin x)Q_\nu(\sin x_0)-P_\nu(\sin x_0)Q_\nu(\sin x)],$$
$$x_0 = \tfrac{1}{2}\pi-\theta_0, \qquad x_1 = \tfrac{1}{2}\pi-\theta_1, \qquad \omega_0(\lambda) = 1. \tag{39}$$

By (11) and (8) we have

$$\left.\begin{aligned} \psi(x,\lambda_n) &= [k_n/\omega_\lambda(\lambda_n)]^{\frac{1}{2}}\phi(x,\lambda_n)\\ k_n &= \frac{(\cos x_1)^{\frac{1}{2}}P_{\nu_n}(\sin x_1)}{(\cos x_0)^{\frac{1}{2}}P_{\nu_n}(\sin x_0)}\end{aligned}\right\}. \tag{40}$$

Calculation of $\omega(\lambda) = \phi\chi_x-\chi\phi_x$ gives

$$\omega(\lambda) = (\cos x_0\cos x_1)^{\frac{1}{2}}[P_\nu(\sin x_0)Q_\nu(\sin x_1)-P_\nu(\sin x_1)Q_\nu(\sin x_0)]. \tag{41}$$

Consequently the roots $\lambda_n = \nu_n(\nu_n+1)$ of $\omega(\lambda) = 0$ satisfy the equation

$$\frac{P_\nu(\sin x_1)}{P_\nu(\sin x_0)} = \frac{Q_\nu(\sin x_1)}{Q_\nu(\sin x_0)}. \tag{42}$$

Therefore $\psi(x, \lambda_n)$ is completely determined.

After transforming (35) with $y = \sin x$ and $u = (\cos x)^{\frac{1}{2}}v$ we apply the finite Sturm–Liouville transform: that is, we multiply the equation by $\psi(x, \lambda_n)$ and integrate between x_0 and x_1. The result is

$$\left.\begin{aligned} (r^2\overline{V}_r)_r - \nu_n(\nu_n+1)\overline{V} &= B_V(r, \lambda_n) \\ B_V(r, \lambda_n) &= [k_n/\omega_\lambda(\lambda_n)]^{\frac{1}{2}}[V_0(r) - k_n^{-1}V_1(r)] \end{aligned}\right\}. \tag{43}$$

Equation (43) is an ordinary differential equation of Euler's type whose solution can be found by variation of parameters. Hence

$$\left.\begin{aligned} \overline{V}(r, \nu_n) &= C_1(\nu_n)r^{\nu_n} + C_2(\nu_n)r^{-\nu_n-1} + F(r, \nu_n) \\ F(r, \nu_n) &= \int^r \frac{B_V(\rho, \lambda_n)}{1+2\nu_n}(r^{\nu_n}\rho^{-\nu_n-1} - r^{-\nu_n-1}\rho^{\nu_n})\, d\rho \end{aligned}\right\}. \tag{44}$$

Let the transforms of $V_3(\theta)$ and $V_4(\theta)$ be $\overline{V}_3(\nu_n)$ and $\overline{V}_4(\nu_n)$; C_1 and C_2 will then be determined from the remaining conditions

$$\left.\begin{aligned} \overline{V} &= \overline{V}_3(\nu_n) \quad (r = r_0) \\ \overline{V} &= \overline{V}_4(\nu_n) \quad (r = r_1) \end{aligned}\right\} \quad (\theta_0 < \theta < \theta_1).$$

This gives two linear equations for C_1 and C_2 whose solutions are

$$\left.\begin{aligned} C_1(\nu_n) &= \frac{[\overline{V}_3(\nu_n) - F(r_0, \nu_n)]r_1^{-\nu_n-1} - [\overline{V}_4(\nu_n) - F(r_1, \nu_n)]r_0^{-\nu_n-1}}{r_0^{\nu_n}r_1^{-\nu_n-1} - r_1^{\nu_n}r_0^{-\nu_n-1}} \\ C_2(\nu_n) &= \frac{[\overline{V}_4(\nu_n) - F(r_1, \nu_n)]r_0^{\nu_n} - [\overline{V}_3(\nu_n) - F(r_0, \nu_n)]r_1^{\nu_n}}{r_0^{\nu_n}r_1^{-\nu_n-1} - r_1^{\nu_n}r_0^{-\nu_n-1}} \end{aligned}\right\}. \tag{45}$$

Hence $\overline{V}(r, \nu_n)$ is completely determined. The inversion theorem (12) now gives

$$V(r, x) = \sum_n \overline{V}(r, \nu_n)\psi(x, \lambda_n), \tag{46}$$

where the summation is extended over all roots of (42).

The analysis given above is formal. It can, however, be made rigorous by showing that the solution satisfies both the differential equation and the boundary conditions.

It may be worth while to remark that with the use of the method of separation of variables the solution of the above problem would have been difficult and lengthy, if not impossible.

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THE GENERALIZATION OF A THEOREM OF L. RÉDEI'S

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Introduction

Our knowledge of the structure of the absolute class group in absolutely cyclic, algebraic number-fields is in general restricted to the class-number formula and the principal-genus theorem. For quadratic fields this theorem determines the number of invariants of the class group in the narrow sense which are divisible by 2. In a series of papers [cf. (**6**), (**7**), (**8**)] L. Rédei† gave, for quadratic fields, a determination of the number of invariants, divisible by 4 and 8. L. Rédei's theorems and some special results by Inaba (**5**), quoted below, have so far been the only instances where an advance beyond the principal-genus theorem has been made.

In the present paper I shall prove a theorem for cyclic fields in general which will contain as special cases L. Rédei's determination of the invariants divisible by 4, and Inaba's results. The aim is to determine certain structural properties of the absolute class group, in the narrow sense, in terms of the rational ideal groups belonging to the given fields. In particular we shall obtain a criterion for the genus number of a field whose degree is a power of a prime l to be equal to the l-class number.

In the same way as the principal-genus theorem for absolutely cyclic fields embodies that information on the structure of their absolute class groups which is deducible from the theory of absolutely Abelian fields, the new theorem represents the application of the theory of fields of at most class two to this problem (**2**), (**3**). A group is said to be *of at most class two* if its commutator subgroup lies in its central, and a field is said to be *of at most class two* if it is self-conjugate and its Galois group is of at most class two. It may be noted that L. Rédei did in fact use fields of at most class two in his work.

Throughout this paper I assume a knowledge of the definitions and results in two earlier papers (**2**), (**3**). The notation here will be often the same as in (**3**). For Theorem 1, also, part of the contents of (**1**) will be required.

Notation

K is a given absolutely cyclic field; P the rational field. The conductor of K is $f(\mathrm{K})$. The absolute Galois group of a field M will be denoted by

† Partly in collaboration with H. Reichardt.

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$\Gamma(M)$, and its relative Galois group over a subfield Λ by $\Gamma(M/\Lambda)$. If τ, δ are elements of a group, then (τ, δ) stands for the commutator $\tau^{-1}\delta\tau\delta^{-1}$, and, if T, Δ are sets of elements in a group, (T, Δ) will stand for the subgroup generated by the commutators (τ, δ) all $\tau \in T$, $\delta \in \Delta$; σ will throughout denote a generator of $\Gamma(K)$.

The group of ideals in K will be denoted by A, and the group of principal ideals, generated by totally positive numbers in K, by I. If Λ is a subfield of K, then the corresponding ideal group in Λ will be indicated by the suffix Λ. Thus I_Λ is the group of principal ideals generated by totally positive numbers in Λ. If M is class field of the field Λ belonging to the ideal group H_Λ, I shall write $M = \Lambda(H_\Lambda)$.

For a given cyclic field Λ I define the group-ring of $\Gamma(\Lambda)$ over the domain of rational integers as an operator domain on A_Λ, writing the operators exponentially. Thus $\mathfrak{a}^\tau$ is the τ-conjugate of the ideal, and $H_\Lambda^\tau = \{\mathfrak{a} \mid \mathfrak{a}^{\tau^{-1}} \in H\}$. Finally, if X is a group, then $\{x\}$ will denote the subgroup generated by the element $x \in X$.

We shall find it convenient to characterize class fields of the rational field and the ideal groups belonging to them by means of Dirichlet's residue characters. We consider two such characters ψ and ψ' as equal if $\psi(a) = \psi'(a)$, whenever both sides of this equation are defined. With this convention the residue characters form an Abelian group. The class field $M = P(H_P)$ then determines a finite group Ψ of residue characters; $(a) \in H_P$, with $a > 0$, if and only if $\psi(a) = 1$, all $\psi \in \Psi$. We shall say that M *belongs to* Ψ. In this way class-field theory sets up a bi-unique correspondence between the absolutely Abelian fields and the finite groups of residue characters. The group Ψ of characters and the Galois group $\Gamma(M)$ of the field M belonging to Ψ are isomorphic, and $\Psi \supset \Psi'$ if and only if $M \supset M'$ for the corresponding fields. For a more formal and detailed statement of the principal theorems of class-field theory in these terms I refer to (**4**). If a character ψ is expressed as a product of characters with prime-power conductors, i.e. $\psi = \prod \psi_i$, where ψ_i has a positive power of a prime p_i as conductor, and $p_i \neq p_j$, for $i \neq j$, then these characters ψ_i will be called the *local components* of ψ. The ramification order of p_i in the field M which belongs to $\{\psi\}$, i.e. the order of the inertia group of p_i in $\Gamma(M)$ is then given by the order of the local component ψ_i.

1. The main object of this paper is the determination of the structure of A *modulo* the subgroup $A^{(1-\sigma)^2}I$. If K is a quadratic field, we note in passing that $A^{(1-\sigma)^2}I = A^{2^r}I$. The problem can immediately be reduced to one on fields of prime-power degree.

THEOREM 1. *If* K *is a cyclic field, then*

$$A/A^{(1-\sigma)^r I} \simeq \prod_{\Lambda} A_\Lambda/A_\Lambda^{(1-\sigma)^r I_\Lambda} \quad (r > 0), \tag{1.1}$$

$$A^{(1-\sigma)^s I}/A^{(1-\sigma)^r I} \simeq \prod_{\Lambda} A_\Lambda^{(1-\sigma)^s I_\Lambda}/A_\Lambda^{(1-\sigma)^r I_\Lambda} \quad (r > s > 0), \tag{1.2}$$

where $\prod_{\Lambda}$ *stands for the direct product,* Λ *running through all maximal subfields of* K *of prime-power degree.*

COROLLARY. *The prime divisors of* $(A : A^{(1-\sigma)^r I})$ *divide the degree* $(\mathrm{K} : \mathrm{P})$.

The proofs of Theorem 1 and its corollary follow from the results in (**1**), and are omitted here.

From now on, unless specially mentioned, I shall assume the degree of K to be a power of a fixed prime l.

In this section I shall also restate, for completeness' sake and for later reference, the classical theorem on the invariants of $A/A^{(1-\sigma)I}$, and some of its corollaries.

THEOREM 2. *Let* K *be a cyclic field of degree* l. *Let* p_i $(i = 1,\ldots, m)$ *be the distinct prime divisors of* $f(\mathrm{K})$, *of ramification orders* l^{ν_i}; *if* $m > 1$, *let* $\nu_i \geqslant \nu_{i+1}$. *Then* $A/A^{(1-\sigma)I}$ *has the invariants* $(l^{\nu_2},\ldots, l^{\nu_m})$.† (*The hypothesis implies* $\nu = \nu_1$.)

I denote the highest power of l dividing $(A : I)$ by h_l. Then Theorem 2 leads to

COROLLARY 1. $$h_l \geqslant l^{\sum_{i=2}^{m} \nu_i}.$$

COROLLARY 2. $h_l = 1$ *if and only if* $f(\mathrm{K})$ *is a prime power.*

2. In this section I shall derive the main theorem on the invariants of $A^{(1-\sigma)I}/A^{(1-\sigma)^2 I}$, and deduce a number of results from it. It may be noted that the invariants of the quotient group $A/A^{(1-\sigma)^2 I}$ can be calculated from those of $A/A^{(1-\sigma)I}$, and $A^{(1-\sigma)I}/A^{(1-\sigma)^2 I}$, using the general structure theorem in (**3**) [cf. Theorem 2]. If l is odd, we have in fact an isomorphism between $A/A^{(1-\sigma)^2 I}$ and the direct product of $A/A^{(1-\sigma)I}$ and $A^{(1-\sigma)I}/A^{(1-\sigma)^2 I}$ [cf. (**3**), Cor. 1, Theorem 2], while, if K is quadratic, we can use the fact that $A^{(1-\sigma)^r I} = A^{2^r I}$.

† Here, and in future, we associate the empty set of invariants with the group of order 1. Throughout the paper p_i and ν_i will have the same meaning as here.

For all ordered pairs of indices (i, j) $(i, j = 1,..., m)$ I define positive integers $[p_i, p_j]$ by

$$0 < [p_i, p_j] \leqslant l^{\nu_i},$$

$$[p_i, p_i] = l^{\nu_i}, \tag{2.1}$$

$$\chi_i(p_j) = \epsilon^{l^{\nu-\nu_i}[p_i,p_j]}, \text{ if } i \neq j.$$

Here ϵ is a fixed primitive l^{ν}th root of unity. K belongs to a cyclic group of characters, of order l^{ν} and conductor $f(\mathrm{K})$, generated by a character

$$\chi = \prod_{i=1}^{m} \chi_i^{y_i}, \qquad (y_i, l) = 1, \qquad y_1 = 1, \tag{2.2}$$

where χ_i is a character of order l^{ν_i} with a power of p_i as conductor $(i = 1,..., m)$.†

Let $\mathrm{K}^* = \mathrm{K}(A^{(1-\sigma)I})$, and let Λ be the maximal non-ramified central class field of K^*. It was proved in (**3**) [cf. Theorem 2] that the Galois group $\Omega = \Gamma(\Lambda)$ is generated by m elements σ_i such that $\{\sigma_i\}$ is the inertia group of a prime divisor of p_i in Λ $(i = 1,..., m)$; and Ω is completely determined by the relations:

$$\sigma_i^{l^{\nu_i}} = 1 \quad (i = 1,..., m), \tag{2.3}$$

$$(\Omega, (\Omega, \Omega)) = 1, \tag{2.4}$$

$$\prod_{j=1}^{m} (\sigma_i, \sigma_j)^{[p_j,p_i]} = 1 \quad (i = 1,..., m). \tag{2.5}$$

K^* is the invariant field of the commutator group (Ω, Ω); this group is Abelian by (2.4), and is generated by the elements (σ_i, σ_j) $(i < j)$ with relations (2.5) and the relations

$$(\sigma_i, \sigma_j)^{p_i} = 1 \quad (i < j) \tag{2.6}$$

following from (2.3).

K is the invariant field of a group Δ generated by (Ω, Ω), and by the elements

$$\gamma_i = \sigma_1^{-s_i}\sigma_i, \qquad s_i = y_i l^{\nu-\nu_i} \qquad (i = 1,..., m) \tag{2.7}$$

(of course $\gamma_1 = 1$).

With this notation we have then

† This definition differs slightly from the one used in (**2**), (**3**).

THEOREM 3. *The invariants of $A^{(1-\sigma)}I/A^{(1-\sigma)^2}I$ are the elementary divisors other than* 1, *in the ring of rational l-integers of the matrix*

$$T = \begin{pmatrix} U \\ L \end{pmatrix}, \qquad U = (u_{ij}), \qquad L = (l_{ij}) \qquad (i, j = 1,\ldots, m),$$

$$u_{ij} = -[p_j, p_i]s_i \text{ if } i \neq j, \qquad u_{ii} = \sum_{j=1}^{m} s_j[p_j, p_i],$$

$$l_{ij} = 0 \text{ if } i \neq j, \qquad l_{11} = 1, \qquad l_{ii} = l^{\nu_i} \qquad (i > 1).$$

Here a rational number is called an l-integer if it can be written as a fraction of integers with denominator prime to l.

Proof. The field $\mathrm{K}(A^{(1-\sigma)^2}I)$ is the maximal non-ramified class field of K which is absolutely of at most class two. It is thus contained in Λ, and in fact is the maximal class field of K in Λ; therefore it is the invariant field of (Δ, Δ). It follows that

$$A^{(1-\sigma)}I/A^{(1-\sigma)^2}I \simeq (\Omega, \Omega)/(\Delta, \Delta).$$

$\Omega/(\Omega, \Omega)$ has the basis $\sigma_1, \gamma_2,\ldots, \gamma_m$, and $\Delta/(\Omega, \Omega)$ is generated by γ_i $(i = 1,\ldots, m)$. Hence $(\Omega, \Omega)/(\Delta, \Delta)$ is generated by elements

$$\omega_i = (\sigma_1, \gamma_i) \quad (i = 1,\ldots, m), \tag{2.8}$$

where

$$\omega_1 = 1, \quad \omega_i^{l^{\nu_i}} = 1 \text{ if } i > 1. \tag{2.9}$$

(2.9) expresses the relations (2.6) in terms of $(\Omega, \Omega)/(\Delta, \Delta)$, while the congruence $(\sigma_i, \sigma_j) \equiv \omega_j^{s_i}\omega_i^{-s_j}(\Delta, \Delta)$ shows that the relations (2.5) lead to

$$\omega_i^{\sum_{j=1}^{m} s_j[p_j,p_i]} \prod_{j=1}^{m} \omega_j^{-[p_j,p_i]s_i} \equiv 1(\Delta, \Delta) \quad (i = 1,\ldots, m). \tag{2.10}$$

The matrix of (2.9) is L, and that of (2.10) is U. Hence $(\Omega, \Omega)/(\Delta, \Delta)$ is isomorphic to the Abelian group generated by elements ω_i $(i = 1,\ldots, m)$ with order matrix T.

We define r as rank U in GF(1). Then we obtain

COROLLARY 1. *The number of invariants of $A^{(1-\sigma)}I/A^{(1-\sigma)^2}I$ is $m-1-r$. If* K *is quadratic, the number of invariants of A/I divisible by* 4 *is $m-1-r$.*†

Proof. The corollary is certainly true by Theorem 3 if we replace r by $r_1 = \operatorname{rank} U_1$, where U_1 is the $(m, m-1)$ matrix obtained from U by leaving out the first column. But $\sum_{j=1}^{m} u_{ij}s_j = 0$ (all i) and $s_1 = 1$; hence $r_1 = r$.

† It is easily verified that the latter result is equivalent to Rédei's matrix criterion (7).

COROLLARY 2. *The genus number is equal to the l-class number h_l, i.e. the inequality in Corollary 1 of Theorem 2 reduces to an equation if and only if $r = m-1$. In particular, if* K *is of prime degree, then $h_l = l^{m-1}$ if and only if $r = m-1$.*

Proof. If $h_l = l^{\sum_{i=2}^{m} \nu_i}$, then by Theorem 1 Corollary, $A^{(1-\sigma)}I = A^{(1-\sigma)^2}I$; thus $r = m-1$. If, conversely, $A^{(1-\sigma)}I = A^{(1-\sigma)^2}I$, then we have $A^{(1-\sigma)}I = A^{(1-\sigma)^n}I$ (all $n \geqslant 1$); hence $A^{(1-\sigma)}I = A^{l(1-\sigma)}I$, and thus $h_l = (A : A^{(1-\sigma)}I)$.

Next let K be of odd prime degree, and denote the number of invariants of A/I which are multiples of l by d_l. Inaba proved in (**5**) that

$$m-1 \leqslant d_l \leqslant (l-1)(m-1).$$

This can now be strengthened to:

THEOREM 4. *If* K *has odd prime degree, then*

$$2(m-1)-r \leqslant d_l \leqslant (l-1)(m-1)-(l-2)r.\dagger$$

This theorem follows from Theorem 3 in the same way as Inaba's inequality can be deduced from Theorem 2. I shall give only a brief outline of the proof; for details see e.g. (**5**).

If $\{\sigma\}$ is cyclic of odd prime degree, and if G denotes the group of ideals whose orders *modulo* I are prime to l, then the class group A/G is the direct product of $m-1$ groups H_i, each of which is generated by the symbolic powers $C_i^{f(1-\sigma)}$ of an ideal class C_i mod G ($i = 1,\ldots, m-1$); the f are polynomials in $(1-\sigma)$ whose rational integral coefficients satisfy the inequality $0 \leqslant c < l$.

We have $(H_i : H_i^{1-\sigma}) = l$ (all i). For $i \leqslant r$, $C_i^{1-\sigma} = C_i^{(1-\sigma)^2}$, and thus $C_i^{1-\sigma}$ is the unit class; H_i is therefore cyclic of degree l. For $i > r$,

$$(H_i^{1-\sigma} : H_i^{(1-\sigma)^2}) = l, \quad H_i/H_i^{(1-\sigma)^2} \simeq H_i/H_i^{(1-\sigma)} \times H_i^{(1-\sigma)}/H_i^{(1-\sigma)^2};$$

therefore $H_i/H_i^{(1-\sigma)^2}$ is of type (l, l).

Finally for all i, $H_i^{(1-\sigma)^{l-1}} \subseteq H_i^l$. Thus H_i, for $i \leqslant r$, has exactly one invariant, and, for $i > r$, not less than 2 and not more than $l-1$ invariants. Summing over all i, we get the theorem.

3. Theorem 3 expresses the invariants of $A^{(1-\sigma)}I/A^{(1-\sigma)^2}I$ in a form in which they can readily be calculated. In the next two sections I shall derive two theorems which determine the structure of this group in a

† Inaba (**5**) first stated this inequality for $m = 2$, $(f(\mathrm{K}), 1) = 1$.

more conceptual manner. The theorem of this section is closely connected with L. Rédei's ideas, and gives for the quadratic case his criterion in terms of discriminant decomposition (**6**), (**7**). It establishes an isomorphism between $A^{(1-\sigma)}I/A^{(1-\sigma)^2}I$ and a group Φ_2 of pairs of residue characters, which will be derived from the given character χ of (2.2) in a natural manner.

L. Rédei proved that, if K is a quadratic field, then the groups A/A^2I, A^2I/A^4I are isomorphic, respectively, to the groups of discriminant decompositions of the first and the second kind. His concept of discriminant decomposition cannot, however, be readily generalized, and will be replaced by the concept of character decomposition; in the quadratic case the two concepts will easily be recognized as equivalent.

I shall use throughout the notation established in §§ 1, 2. In particular, the character χ of (2.2) is again a generator of the group to which K belongs.

An ordered pair of residue characters $(\psi, \chi\psi^{-1})$, such that the local components of ψ can be expressed in terms of the local components of χ is said to be a *χ-decomposition*, or a *χ-decomposition of the first kind*, and ψ is said to *determine* the decomposition. The field M belonging to the character group $\{\psi, \chi\psi^{-1}\} = \{\psi, \chi\}$ will also be said to *belong to the* χ-decomposition $(\psi, \chi\psi^{-1})$. It is evident then that ψ determines a χ-decomposition if and only if ψ lies in the character group Ψ which corresponds to $\mathrm{K}(A^{(1-\sigma)}I)$.

Two χ-decompositions $(\psi, \chi\psi^{-1})$ and $(\psi_1, \chi\psi_1^{-1})$ will be called *equivalent*, or belonging to the same χ-decomposition class, if $\psi_1 = \chi^r\psi$ with some r, i.e. if $\psi_1 \equiv \psi(\{\chi\})$. This is evidently a proper equivalence relation. To every such class there belongs a unique field M.

We can define a multiplication of two χ-decomposition classes (ψ, ψ') and (ψ_1, ψ_1') by

$$(\psi, \psi')(\psi_1, \psi_1') = (\psi\psi_1, \psi'\psi_1'\chi^{-1}), \tag{3.1}$$

where it is obvious that the choice of representatives is irrelevant. The multiplication of χ-decomposition classes is thus determined by the multiplication of their determining character co-sets $\mathrm{mod}\{\chi\}$, and these classes form an Abelian group isomorphic to $\Psi/\{\chi\}$, with identity $(1, \chi)$.

Let M belong to the χ-decomposition class $(\psi, \chi\psi^{-1})$. Then this class is said to be a χ-decomposition class *of the second kind* if, for all p_i dividing $f(\mathrm{K})$, the order of the decomposition group of p_i in $\Gamma(\mathrm{M})$ does not exceed the degree of K. If, in particular, $v_i = v$ (all i), then $(\psi, \chi\psi^{-1})$ is of the second kind if and only if all prime-divisors of $f(\mathrm{K})$ are of the first degree in M.

We can now state:

THEOREM 5. *Let the cyclic field* K *belong to the group* $\{\chi\}$ *of residue characters. A pair of characters* $(\psi, \chi\psi^{-1})$ *defines a* χ*-decomposition class if and only if the Abelian field* M *belonging to* $\{\psi, \chi\}$ *is relatively cyclic and non-ramified over* K. *The* χ*-decomposition classes form an Abelian group* $\Phi_1 \simeq A/A^{(1-\sigma)}I$. *The relative degree of the field* M *belonging to a* χ*-decomposition class* $(\psi, \chi\psi^{-1})$ *is equal to the group order of* $(\psi, \chi\psi^{-1})$.

A χ*-decomposition class* $(\psi, \chi\psi^{-1})$ *of order* t *is of the second kind if and only if there exists a non-ramified class field* M^* *of* K, *of relative degree* t^2, *which is absolutely self-conjugate and of at most class two, and which is cyclic over the field* M *belonging to* $(\psi, \chi\psi^{-1})$, *and contains this field as its maximal absolutely Abelian subfield.*

The χ*-decomposition classes of the second kind form a group*

$$\Phi_2 \simeq A^{(1-\sigma)}I/A^{(1-\sigma)^2}I.$$

Remark. In this theorem it is not necessary to assume that K has prime-power degree. The preceding definitions of χ-decomposition classes can readily be extended to the general case. It is, however, sufficient to prove the theorem under the restriction to prime-power degrees.

Proof. Let M be the field belonging to $\{\psi, \chi\}$; M is certainly cyclic over K. M is non-ramified over K if and only if $\psi \in \Psi$, i.e. if and only if $(\psi, \chi\psi^{-1})$ is a χ-decomposition.

We assume from now on that $(\psi, \chi\psi^{-1})$ is in fact a χ-decomposition. Without loss of generality we may put

$$\psi = \prod_{i=1}^{m} \chi^{x_i}, \qquad x_1 = l^\nu. \tag{3.2}$$

Writing
$$t_i = l^{\nu-\nu_i}x_i \quad (i = 1,\ldots, m), \tag{3.3}$$

we have
$$t_1 \equiv 0\,(l^\nu), \qquad l^{\nu_i}t_i \equiv 0\,(l^\nu) \quad (i = 2,\ldots, m). \tag{3.4}$$

Conversely, if (3.2)–(3.4) hold, then $(\psi, \chi\psi^{-1})$ is a χ-decomposition, and is uniquely determined in its class by the condition $t_1 \equiv 0\,(l^\nu)$. It now follows trivially from (3.4) that Φ has the same invariants as $A/A^{(1-\sigma)}I$.

In order to show that the χ-decomposition classes of the second kind form a group $\Phi_2 \simeq A^{(1-\sigma)}I/A^{(1-\sigma)^2}I$, we shall deduce a system of homogeneous linear congruences with matrix T, satisfied by the t_i if and only if $(\psi, \chi\psi^{-1})$ is of the second kind, ψ being defined by (3.2), (3.3).

Let l^λ be the order of ψ. Then

$$t_i \equiv 0\,(l^{\nu-\lambda}) \text{ (all } i); \qquad \exists\, q\ (2 \leqslant q \leqslant m), \quad t_q \not\equiv 0\,(l^{\nu-\lambda+1}). \tag{3.5}$$

Let M be the invariant field of the sub-group Θ of $\Omega = \Gamma(\Lambda)$. The decomposition group Π_i of p_i in Ω/Θ is generated by the elements

$$\sigma_i \equiv \sigma_1^{s_i} \gamma_q^{t_i/t_q}(\Theta),$$

$$\delta = \sigma_1^{\sum_{j=1}^{m} s_j[p_j,p_i]} \gamma_q^{\sum_{j=1}^{m} (t_j/t_q)[p_j,p_i]} \equiv \prod_{j=1}^{m} \sigma_j^{[p_j,p_i]}(\Theta).$$

Order $\Pi_i \leqslant l^\nu$ if and only if $\delta^{l^{\nu-\nu_i}} \equiv 1(\Theta)$; i.e. if and only if $\exists\ r$ with

$$l^{\nu-\nu_i} \sum_{j=1}^{m} s_j[p_j,p_i] \equiv r s_i, \qquad l^{\nu-\nu_i} \sum_{j=1}^{m} t_j[p_j,p_i] \equiv r t_i \ (l^\nu). \tag{3.6}$$

By (2.7), (3.3), however, (3.6) is equivalent to

$$\sum_{j=1}^{m} t_j[p_j,p_i]s_i - s_j[p_j,p_i]t_i \equiv 0\,(l^\nu). \tag{3.7}$$

Then (3.4) and (3.7), for $i = 1,\ldots, m$, constitute the required system of congruences.

To complete the proof, we observe that (Ω,Ω) is generated by (Θ,Ω) and the element $\omega_q = (\sigma_1, \gamma_q)$, and that the cyclic quotient group $(\Omega,\Omega)/(\Theta,\Omega)$ is by (2.9), (2.10) determined uniquely by the relations

$$\omega_q^{l^\lambda} \equiv 1((\Theta,\Omega)),$$

$$\omega_q^{\sum_{j=1}^{m} s_j[p_j,p_i](t_i/t_q)-(t_j/t_q)[p_j,p_i]s_i} \equiv 1((\Theta,\Omega)) \quad (i = 1,\ldots, m).$$

Thus $(\Omega,\Omega):(\Theta,\Omega)$ attains its maximum value l^λ if and only if (3.7) holds. It follows that the maximal non-ramified central class field $\overline{\mathrm{M}}$ of M is cyclic of at most degree l^λ over $\mathrm{K}^* = \mathrm{K}(A^{(1-\sigma)}I)$, and that the degree l^λ is attained if and only if $(\psi, \chi\psi^{-1})$ is of the second kind.

If then $(\overline{\mathrm{M}}:\mathrm{K}) = l^\lambda$, one easily deduces from the structural properties of Ω that $\exists$ a self-conjugate field $\mathrm{M}^*_\cap$, with $\mathrm{M}^*\mathrm{K}^* = \overline{\mathrm{M}}$, $\mathrm{M}^*_\cap \mathrm{K}^* = \mathrm{M}$. But then M^*/M is cyclic of degree l^λ and non-ramified, and M is indeed the maximal absolutely Abelian sub-field of M^*.

Conversely, if M^* is a field with the properties postulated in the theorem, then $\overline{\mathrm{M}} = \mathrm{M}^*\mathrm{K}^*$ is a non-ramified central class field of M such that $(\overline{\mathrm{M}}:\mathrm{K}^*) = l^\lambda$, and hence $(\psi, \chi\psi^{-1})$ is of the second kind.

4. We assume now K to have prime degree l. I denote by P the group of rational numbers generated by the prime divisors of $f(\mathrm{K})$. Thus $a \in P$ if and only if

$$a = \prod_{j=1}^{m} p_j^{\nu_j}. \tag{4.1}$$

In P, I consider the subgroup P_κ of total norm residues of K, i.e. the subgroup of numbers a characterized by

$$\left(\frac{a, \mathrm{K}}{p_i}\right) = 1 \quad (\text{all } i), \tag{4.2}$$

where $\left(\frac{x, \mathrm{K}}{p}\right)$ is Hasse's norm-residue symbol. P_κ contains the group P^l of the lth power in P.

Now (4.2) can be expressed in terms of congruence relations in the group Ω/Δ by observing that

$$\left(\frac{a, \mathrm{K}}{p_i}\right) \equiv \sigma_i^{\sum_{j=1}^{m}[p_i,p_j]t_j} \prod_{j=1}^{m} \sigma_j^{-[p_j,p_i]t_i}$$

$$\equiv \sigma_1^{\sum_{j=1}^{m} s_i[p_i,p_j]t_j - \sum_{j=1}^{m} s_j[p_j,p_i]t_i} (\Delta).$$

Thus (4.2) is equivalent to

$$\sum_{j=1}^{m} s_j[p_j, p_i]t_i - \sum_{j=1}^{m} s_i[p_i, p_j]t_j = 0\,(l) \quad (i = 1,\ldots, m). \tag{4.3}$$

I now define the matrix $U^* = (u_{ij}^*)$ $(i = 1,\ldots, m)$ by

$$u_{ij}^* = -s_j[p_j, p_i] \quad (i \neq j), \qquad u_{ii}^* = \sum_{j=1}^{m} s_j[p_j, p_i].$$

This matrix I consider as a matrix over GF(l); I denote its rank by r^*. Then $m - r^*$ is the number of invariants of P_κ/P^l.

On the other hand $(s_i, l) = 1$ if $(\mathrm{K}:\mathrm{P}) = l$. Therefore the matrix $D = (\delta_{ij} s_i)$ $(i, j = 1,\ldots, m)$, where δ_{ij} is the Kronecker symbol, is non-singular over GF(l); but $D^{-1}UD = U^*$, where U is the matrix with rank r defined in Theorem 3. Hence, by Corollary 1 of Theorem 3, we have Theorem 6, which is completed by a well-known result on $A/A^{(1-\sigma)}I$,

THEOREM 6. *Let* K *be of prime degree* l. *Then*

number of invariants of P/P^l = (*number of invariants of* $A/A^{(1-\sigma)}I$)+1,

number of invariants of P_κ/P^l

= (*number of invariants of* $A^{(1-\sigma)}I/A^{(1-\sigma)^2}I$)+1.

It may be of interest that Theorem 6 can be proved in an elementary manner, without recourse to the theory of fields of at most class two.

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A NOTE ON THE CLASS FIELD TOWER

By A. FRÖHLICH (*Keele*)

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LET K be an absolutely cyclic, algebraic number field of prime degree l. I define by induction extension fields K_n of K, for all $n \geqslant 0$, as follows. $K_0 = K$; K_{n+1} is the maximal non-ramified class field of K_n whose relative degree is a power of l.† The number of distinct rational prime divisors of the discriminant of K will be denoted by $m(K)$. Then one can pose the question:

If $m(K) = m$ is a prescribed integer, what can be said about the number of distinct fields K_n?

The only known results, relevant to this question, are

(i) $K_1 = K_0$, *if* $m \leqslant 1$;

(ii) $K_1 \neq K_0$, *if* $m > 1$.

In this note I shall prove

(iii) $K_2 \neq K_1$, *if* $m > 3$;

(iv) (iii) *is best-possible in the sense that, for prescribed* l *and for* $m = 2, 3$, *there exists a field* K *of degree* l, *with exactly* m *prime divisors of the discriminant, such that* $K_1 = K_2$.

These results go for the first time beyond the classical propositions (i), (ii), and together with these seem to point to some general law of the form

(v) $K_{n+1} \neq K_n$, *if* $m > g(n)$;

(vi) *for any given* l, *and any* $m \leqslant g(n)$, $\exists$ K, *of degree* l, *with* $m(K) = m$, *such that* $K_{n+1} = K_n$.

A. Scholz in (**4**) has constructed, for given n and l, cyclic fields of degree l such that $K_{n+1} \neq K_n$. His approach, however, differs from mine in so far as he aims at existence theorems and his construction depends on the choice of the discriminant. It is of interest to our question, though, that his $m(K)$ increases with n.

In the same way as (i), (ii) follow from the arithmetical theory of absolutely Abelian fields, (iii) and (iv) follow from the arithmetical theory of fields of at most class two [cf. (**1**), (**2**)]. It seems plausible to expect that any further advance of the arithmetical theory of algebraic number

† 'Non-ramified' refers here only to the finite prime divisors.

fields on these lines will lead to further results in the direction of (v) and (vi).

(iii) is the corollary of a theorem which establishes a sufficient condition for $K_2 \neq K_1$ in terms of rational congruence relations between the discriminant divisors. I shall use here the results of (**2**), and the results and notations of the preceding paper (**3**).

As in (**3**) § 2, K^* is the maximal non-ramified class field of K, which is still absolutely Abelian, and Λ is the maximal, non-ramified central class field of K^*. I consider again the Galois group Ω of Λ. The maximal class field $\overline{K}$ of K in Λ is the invariant field of the group (Δ, Δ). If

$$\Lambda \neq \overline{K}, \tag{1}$$

then $\Lambda K_1 \neq \overline{K}K_1 = K_1$. But $\Lambda/\overline{K}$ and $\Lambda K_1/K_1$ are both relatively Abelian, and non-ramified, and therefore

$$K_2 \neq K_1. \tag{2}$$

If now $\bar{d}$ denotes the number of invariants of (Ω, Ω) and d the number of invariants of $(\Omega, \Omega)/(\Delta, \Delta)$, then (1) holds if and only if

$$d < \bar{d}. \tag{3}$$

I choose as basis of $\Omega/(\Omega, \Omega)$ the elements

$$\sigma_1, \quad \gamma_i = \sigma_1^{-s_i}\sigma_i \quad (i = 2,..., m), \tag{4}$$

where the γ_i have the same meaning as in (**3**), and where $(s_i, l) = 1$. Then (Ω, Ω) is generated by the elements

$$\omega_i = (\sigma_1, \gamma_i) \quad (i = 2,..., m), \quad (\gamma_i, \gamma_j) \quad (2 \leqslant i < j \leqslant m). \tag{5}$$

These elements have order at most l; (Ω, Ω) is then completely determined by the relations

$$\prod_{j=1}^{m} (\sigma_i, \sigma_j)^{-[p_j, p_i]} = 1,$$

i.e., putting formally

$$\omega_1 = 1, \qquad (\gamma_1, \gamma_i) = (\gamma_i, \gamma_1) = 1 \quad \text{(for all } i\text{)},$$

by the relations

$$\prod_{j=1}^{m} (\gamma_i, \gamma_j)^{-[p_j, p_i]} \prod_{j=1}^{m} \omega_j^{-[p_j, p_i]s_i} \omega_i^{\sum_{j=1}^{m} s_j[p_j, p_i]} = 1 \quad (i = 1,..., m). \tag{6}$$

I define two matrices $U = (u_{ij})$, $V = (v_{ls})$ in GF(l). In U, $i = 1,..., m$, $j = 2,..., m$, and

$$u_{ij} = -[p_j, p_i]s_i \quad (i \neq j), \qquad u_{ii} = \sum_{j=1}^{m} s_j[p_j, p_i].\dagger$$

† The notation differs here from that in (**3**).

In V, $t = 1,\ldots, m$, while s runs through all pairs of indices (i,j), with $2 \leqslant i < j \leqslant m$, and

$$v_{ts} = -[p_j, p_i], \text{ if } s = (i,j),\ t = i,$$
$$v_{ts} = [p_i, p_j], \quad \text{if } s = (i,j),\ t = j,$$
$$v_{ts} = 0, \text{ otherwise.}$$

Finally let $W = (UV)$, i.e. W is an m by $\binom{m}{2}$ matrix. The ranks† of U, V, W are denoted by u, v, w. Then as in (**3**) [Cor. 1, Theorem 3]

$$d = m-1-u, \tag{7}$$

while by (6)

$$\bar{d} = \binom{m}{2} - w. \tag{8}$$

Hence, by (1)–(3), we have

THEOREM. *If*

$$\binom{m}{2} - (m-1) > w-u, \tag{9}$$

then $\mathrm{K}_2 \neq \mathrm{K}_1$.

We now observe that $w \leqslant m$; hence (9) is certainly satisfied if $m > 4$. For $m = 4$, (9) becomes

$$3+u > w. \tag{9a}$$

This again certainly holds if $u > 1$. If $u = 0$, then $[p_i, p_j] = 0$, for all i, j (as elements in GF(l)). Hence $w = 0 < 3$. If $u = 1$, we can without loss of generality assume $[p_2, p_1] \neq 0$. Then, for $k = 2, 3, 4$, $\exists\ \lambda_k$, such that $[p_i, p_k] = \lambda_k [p_i, p_1]$. Substituting this equation in the matrix V, we verify that $v \leqslant 2$. But $w \leqslant u+v$. Hence $w \leqslant 3 < 3+u = 4$, and (iii) is proved.

In order to establish (iv) I shall show that for given l, and for $m = 2, 3$, there exists a cyclic field K of degree l with exactly m discriminant divisors, such that the field $\mathrm{K}^* = \mathrm{K}(A^{(1-\sigma)}I)$ has l-class number 1. Then $\mathrm{K}^* = \mathrm{K}_1$, and hence $\mathrm{K}_2 = \mathrm{K}_1$.

Let $p_i \equiv 1(l)$ $(i = 1,\ldots, m)$ and let K be a cyclic field of degree l whose discriminant has exactly the primes p_i as its prime divisors. If $m = 2$, let p_2 be an lth non-residue to modulus p_1. If $m = 3$, let p_{i+1} be an lth non-residue to modulus p_i $(i = 1, 2, 3)$, where we put $p_1 = p_4$, and let p_1 be an lth residue to modulus p_2. Then it follows from (**2**), Theorem 3, that K^* has l-class number 1.

That primes with the required properties do in fact exist is well known: it can be proved, e.g. by considering the decomposition of primes in

† If the number of columns of a matrix is zero, I shall define its rank as zero.

appropriate algebraic number fields and using Tchebotareff's density theorem.

The restriction to fields of prime degree is not a necessary one. Similar results hold for cyclic fields of prime-power degree. To bring out the main points of this note it is, however, sufficient to consider only this special case.

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ON THE BOUND OF REGULARITY OF THE SOLUTIONS OF ANALYTIC DIFFERENTIAL EQUATIONS OF FIRST ORDER

By A. WINTNER (*Johns Hopkins*)

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1. Let $f(z, w)$ be a function regular on a neighbourhood of $(z, w) = (0, 0)$, say on

$$|z| < a, \quad |w| < b. \tag{1}$$

If a, b are replaced by somewhat smaller numbers, then it can be assumed that f is bounded on (1), say

$$|f(z, w)| \leqslant M \quad \text{on (1).} \tag{2}$$

By keeping $M = M(a, b)$ fixed and then decreasing the value of b, it can also be assumed that

$$a > b/M. \tag{3}$$

The trivial case $M = 0$ will always be excluded.

According to Cauchy, there exists on some neighbourhood of $z = 0$ one and only one regular function $w(z) = w_f(z)$ satisfying the differential equation and initial condition

$$w' \equiv dw/dz = f(z, w) \quad \text{and} \quad w(0) = 0. \tag{4}$$

In addition, if $r = r_f$ denotes the radius of the largest circle about $z = 0$ within which $w = w_f(z)$ has no singular point z (so that $0 < r_f \leqslant \infty$ for every f, and $r_f = \infty$ only for certain functions f) and if a normalization is so chosen that $f(z, w)$ is regular on (1) and satisfies (2), then Cauchy's method of majorants supplies for r_f a positive lower bound, say r^*, which depends only on a, b, M but not on f.

It was pointed out in (3) that, whilst various efforts have been made to improve on Cauchy's r^*, the *best* value of r^* (which *cannot* be obtained by *any* majorant method; cf. (16) below) is supplied by the method of successive approximations. For, if (3) is assumed, then, on the one hand, the latter method assures that $r_f \geqslant b/M$ and, on the other hand, it is possible to choose an $f = f_\epsilon$ satisfying $r_f < (1+\epsilon)b/M$, if $\epsilon > 0$ is given arbitrarily [cf. (15) below].

Accordingly, the best possible value of r^* is b/M with the assumption (3), and hence $\min(a, b/M)$ without the assumption (3). (The latter conclusion, that in which (3) is not assumed, is trivial if it is observed that

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(4) is equivalent to

$$w(z) = \int_0^z f(Z)\,dZ \quad (|z| < a) \tag{5}$$

when $f(z, w)$ is a function $f(z)$ of z alone and that the function (5) is singular at $z = a$ when its derivative $f(z)$ is.) It turns out, however, that this determination of the true value of r^* does not contain everything that can *always* be said on the 'nearest singularities' of the solutions $w = w_f(z)$ of the problem (4) under the assumptions (2) and (3).

2. It will always be assumed that $f(z, w)$ is regular on (1) and that (2) and (3) are satisfied. Then the successive approximations, which supply for (4) the existence of its (regular) solution $w(z)$ in the circle $|z| < b/M$ (at least), also show that

$$|w(z)| < b \quad \text{if } |z| < b/M. \tag{6}$$

The improvements, referred to above, of the facts collected before will centre around the following improvement of (6):

$$\sup_{|z|<b/M} |w(z)| < b \tag{7}$$

unless $$w(z) = cz \quad (c \text{ constant}, \ |c| = |f(0, 0)|). \tag{8}$$

First, if the case (8) is excluded, it follows from (6) and from Schwarz's lemma (which, in view of the second of the relations (4), is applicable) that

$$|w(z)| < M|z| \quad \text{if } 0 < |z| < b/M. \tag{9}$$

But more than (9) is true since, on (1),

$$|f(z, w)| < M. \tag{10}$$

In fact, if (2) fails to imply (10), then $f(z, w)$ is a constant, and hence (4) leads to (8). Since (2), (3), and (10) imply that $|f(z, w(z))| < M$ if $|z| < b/M$, it follows from (4) that

$$|w'(z)| < M \quad \text{if } |z| < b/M. \tag{11}$$

If $w'(z)$ is integrated along the rectilinear path joining $z = 0$ to $z = re^{i\phi}$, where $0 < r < 1$, it follows from (11), where $w(0) = 0$, that

$$\sup_{0\le r<b/M} |w(re^{i\phi})| < b \tag{12}$$

holds for every *fixed* ϕ.

By using (11) in a slightly different way it is possible to establish not only (12) but also the improvement (7) of (12). This will not however be done, since, by using (12) alone (for fixed ϕ), it will be possible to obtain a result containing (7) as a by-product.

3. It will be shown that to every ϕ there belongs an $\epsilon = \epsilon_\phi$ in such a way that the function $w(z)$ (which, being regular in the circle $|z| < b/M$, is regular on the segment $0 < r < b/M$ of the half-line $\arg z = \phi$) is regular on the segment $0 < r < (1+\epsilon_\phi)b/M$ of the half-line $\arg z = \phi$ In view of the Heine–Borel theorem, this will imply that a positive ϵ_ϕ can be chosen independent of ϕ, which means that $w(z)$ must be regular in a circle $|z| < R$ of some radius $R > b/M$. In view of (12) and the maximum principle, (7) will of course follow as a corollary.

In the proof for the existence of an $\epsilon_\phi > 0$, it can be assumed that $\phi = 0$. Then (12) simplifies to

$$|w(r)| < d \quad (d \text{ constant}, 0 < d < b), \qquad r \to b/M \quad (r < b/M). \tag{13}$$

The assertion is that $w(z)$ must be regular on a segment containing the segment $0 \leqslant z < b/M$ in its interior. The existence of some such (larger) segment will be concluded by applying an argument due to Painlevé [(**2**) 23; for a theorem corresponding to that of Painlevé in the real case and depending on the same argument, cf. (**4**)].

Put $z_k = (1-1/k)b/M$ and denote the value of $w(z)$ at $z = z_k$ by w_k. Then (6) shows that the (z, w)-domain

$$|z| < a, \quad |w-w_k| < b_k \quad (b_k = b-|w_k|) \tag{14_k}$$

is contained in the domain (1). Hence (10) is applicable on (14_k). Consequently, if k is large enough, then that solution $w(z)$ of the differential equation $w' = f(z, w)$ which is defined by the initial condition $w(z_k) = w_k$ is regular on the circle $|z-z_k| < b_k/M$ (this follows from the above-mentioned application of the method of successive approximations). But, if k is large enough, then $|w_k| < d < b$, by (13). Hence the number b_k, having been defined to be $b-|w_k|$, exceeds the positive lower bound $b-d$. Consequently, the function $w(z)$ is regular in the circle

$$|z-z_k| < (b-d)/M,$$

one of fixed radius but of variable centre (z_k). Since $z_k \to b/M$ as $k \to \infty$, it follows that $w(z)$ is regular on the segment $0 \leqslant z < (\epsilon_0+b)/M$, where $\epsilon_0 = b-d > 0$.

4. This proves (ii) and (iii) and hence the first part of (i) in the following theorem:

THEOREM. *If $f(z, w)$ is regular on* (1) *and satisfies* (2) *with* (3) *(but is otherwise unspecified), then*

(i) *the solution $w(z)$ of* (4), *though regular in the circle $|z| < b/M$, can possess a singularity in the circle $|z| < (1+\epsilon^*)b/M$ if $\epsilon^* > 0$ is any absolute constant*;

(ii) *the solution* $w(z)$ *cannot become singular on the circle* $|z| = b/M$, *i.e. there exists an* $\epsilon = \epsilon(f) > 0$ *such that* $w(z)$ *is regular for*

$$|z| < (1+\epsilon)b/M;$$

(iii) *if* $\epsilon = \epsilon(f) > 0$ *is sufficiently small, then either* $|w(z)| < b$ *holds for* $|z| < (1+\epsilon)b/M$ *or* $w(z)$ *is of the form* (8).

The remaining assertion of (i), that concerning the non-existence of an absolute $\epsilon^* > 0$, can be seen as follows [cf. (3)]. Let

$$f(z, w) \equiv f(w) = (\tfrac{1}{2}+\tfrac{1}{2}w)^{1/n}, \tag{15}$$

where $f(0) = 1$ and $n = 2, 3, \ldots$. If $b = 1$, $M = 1$, and $a > 1$, then the function (15) is regular on (1) and satisfies (2) with (3). Hence the solution $w(z)$ of the case (15) of (4) is regular in the circle $|z| < 1\ (= b/M)$, and even on the boundary of this circle, for every n. But, if $\epsilon > 0$ is given, then the $n = n_\epsilon$ in (15) can be chosen in such a way that the corresponding $w(z)$ has a singularity within the circle $|z| < 1+\epsilon$. In fact, the solution $w(z)$ of the case (15) of (4) is

$$w(z) = -1+(1+z/c_n)^m,$$

where $$c_n = 2^{1/n}m^{-1}, \qquad m = n/(n-1),$$

which shows that $w(z)$ becomes singular at $z = -c_n$ and that $1 < |c_n| \to 1$ as $n \to \infty$.

5. Cauchy's method of majorants, the one which forms the basis of his result on the existence of an absolute $r^* = r^*(a, b, M) > 0$ (mentioned after (4) above), is built on the following principle of comparison.

(*A*) Let $|a_{ik}| \leqslant b_{ik}$ hold for the respective coefficients of two power series

$$f(z, w) = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} a_{ik}z^i w^k, \qquad g(z, w) = \sum_{i=0}^{\infty}\sum_{k=0}^{\infty} b_{ik}z^i w^k,$$

and let r_h denote the radius of convergence of the power series which represents the solution $w(z)$ of (4_h), where (4_h) refers to the case $f = h(z, w)$ of (4). Then $r_f \geqslant r_g$.

In order to apply this comparison theorem (*A*), Cauchy assumes that a given function $f(z, w)$ is regular on (1) and satisfies (2). Then the first assumption of (*A*) is satisfied by the choice $b_{ik} = M/a^i b^k$. But the corresponding $g(z, w)$ is $M\lambda(z/a)\lambda(z/b)$, where $\lambda(s) = 1/(1-s)$ and $|s| < 1$, and this $g(z, w)$ depends only on the three numbers a, b, M. Hence the existence of an 'absolute' $r^* > 0$ follows from $r_f \geqslant r_g$ if $r_g > 0$ is true. Finally, $r_g > 0$ follows by calculating the solution of (4_g) explicitly, which is possible since the differential equation is now of the type $W(w)\,dw = Z(z)\,dz$.

The roughness of the estimate which this r_g supplies for the 'absolute' $r^* = r^*(a, b, M)$ is not caused by Cauchy's choice $|a_{ik}| \leqslant M/a^i b^k$ of the coefficients b_{ik} alone, but simply by the application of the principle (A) in *any* form. In other words, *no* application of the method of majorants can prove a best-possible result for r^*. In fact, the best-possible $g(z, w)$ belonging to a fixed $f(z, w)$ results by placing $b_{ik} = |a_{ik}|$, i.e. by choosing $g = f^*$, where

$$f^*(z, w) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} |a_{ik}| z^i w^k \quad \text{if} \quad f(z, w) = \sum_{i=0}^{\infty} \sum_{k=0}^{\infty} a_{ik} z^i w^k. \quad (16)$$

But, if $f(z, w)$ is regular on (1) and satisfies (2), then the numbers a, b occurring in (1) must, in general, be replaced by $a' = \frac{1}{3}a$, $b' = \frac{1}{3}b$ before it becomes true that

$$|f^*(z, w)| \leqslant M \quad \text{if} \quad |z| \leqslant a', \ |w| \leqslant b' \qquad (2^*)$$

[Bohr; cf., e.g., (1) 26–9].

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NOTE ON CROSS-SECTIONS OVER CW-COMPLEXES

By WILLIAM BARCUS (*Oxford*)

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1. Introduction

STEENROD, in his book on fibre bundles [(**4**), part III], considers the following situation: let X be a finite cell complex in which each cell is contractible on itself to a point, $\mathscr{B}$ a fibre bundle over X with fibre Y, and g a cross-section over the n-section X^n. Then, if σ is a closed $(n+1)$-cell of X, $g \mid \dot{\sigma}$ determines a map of S^n into B_σ, that part of the bundle space over σ. Since σ is contractible, B_σ contracts to Y_x, $x \in \sigma$; and, if Y is assumed to be n-simple, $g \mid \dot{\sigma}$ determines an element $\alpha_\sigma \in \pi_n(Y_x)$; g can be extended to a cross-section over σ if and only if $\alpha_\sigma = 0$. This basic construction is used to assign to g a cocycle c in a certain cochain group having coefficients in a bundle such that g has an extension to a cross-section over X^{n+1} if and only if $c = 0$. Steenrod also shows that $g \mid X^{n-1}$ has an extension to a cross-section over X^{n+1} if and only if the cohomology class of c is 0.

This note gives a theory for the case of fibre spaces over CW-complexes (**5**), with no restrictions on the fibre. These hypotheses clearly include the case treated by Steenrod; but even then the results are somewhat simpler, in that the coefficients are taken from a group instead of a bundle of coefficients.

The term 'fibre space' is used in the sense of Serre (**3**): a *fibre space* is a triple (E, p, A), where E and A are topological spaces and p is a map of E onto A such that the homotopy lifting theorem holds for maps of finite polyhedra into E. The lifting can be strengthened by the following theorem, due to James and Whitehead (**2**). The corollary will be used without further comment.

[1.1] THEOREM. *Let (E, p, A) be a fibre space, and let R be a CW-complex, S a subcomplex. Suppose given a map $\theta\colon R \to E$, a homotopy H of $p\theta$, and a homotopy $\tilde{H}'$ of $\theta \mid S$ such that $p\tilde{H}' = H \mid S \times I$. Then H can be lifted to a homotopy of θ agreeing with $\tilde{H}'$ on $S \times I$.*

[1.2] COROLLARY. *Suppose given a map $\theta\colon R \to E$ and a homotopy H of $p\theta$. Then, if H is* rel S, *it can be lifted to a homotopy of θ which is also* rel S.

Throughout the note let (E, p, K) be a fixed fibre space, where K is a

connected CW-complex and E is a pathwise connected space. Let L be a subcomplex of K, which may be empty unless otherwise stated, and denote by $\bar{K}^n$ the complex $L \cup K^n$. We shall use the system of chain groups

$$C_n(\tilde{K}, \tilde{L}) = H_n(\tilde{K}^n \cup \tilde{L}, \tilde{K}^{n-1} \cup \tilde{L}),$$

where $\tilde{K}$ is the universal covering complex of K and $\tilde{L}$ is the subcomplex of $\tilde{K}$ lying over L.

Let $x_0 \in K^0$ be a fixed base point. Then $\pi_1(K, x_0)$ is a group of operators

(*a*) on $C_n(\tilde{K}, \tilde{L})$, induced by the covering transformations of $\tilde{K}$,

(*b*) on $\pi_n(\bar{K}^n, \bar{K}^{n-1}, x_0)$ for $n \geqslant 3$, acting through $\pi_1(K, x_0) \approx \pi_1(\bar{K}^{n-1}, x_0)$,

(*c*) on $\pi_2(\bar{K}^2, \bar{K}^1, x_0)/C$, where C is the commutator subgroup, as follows.

Let $\eta\colon \pi_2 \to \pi_2/C$ be the projection, and let $t_*\colon \pi_1(\bar{K}^1, x_0) \to \pi_1(K, x_0)$ be the injection (which is onto). If $\alpha \in \pi_2(\bar{K}^2, \bar{K}^1, x_0)$, $\xi \in \pi_1(K, x_0)$, then $\xi(\eta(\alpha)) = \eta(\zeta(\alpha))$, where $\zeta \in t_*^{-1}(\xi)$.

The methods of (**6**) § 12 can be used to show that $C_n(\tilde{K}, \tilde{L})$ is naturally† operator-isomorphic to $\pi_n(\bar{K}^n, \bar{K}^{n-1}, x_0)$ for $n \geqslant 3$, and to $\pi_2(\bar{K}^2, \bar{K}^1, x_0)/C$ for $n = 2$; and that the following diagram commutes:

$$\begin{array}{ccccccc}
\ldots \to C_n(\tilde{K}, \tilde{L}) & \xrightarrow{\partial_n} & C_{n-1}(\tilde{K}, \tilde{L}) & \xrightarrow{\partial_{n-1}} & \ldots & \xrightarrow{\partial_3} & C_2(\tilde{K}, \tilde{L}) \\
\uparrow h & & \uparrow h & & & & \uparrow h \\
\ldots \to \pi_n(\bar{K}^n, \bar{K}^{n-1}, x_0) & \xrightarrow{\gamma_n} & \pi_{n-1}(\bar{K}^{n-1}, \bar{K}^{n-2}, x_0) & \xrightarrow{\gamma_{n-1}} & \ldots & \xrightarrow{\gamma_3} & \pi_2(\bar{K}^2, \bar{K}^1, x_0)/C,
\end{array}$$

where h is the natural operator isomorphism, $\gamma_n = j_{n-1}\partial_n$ for $n > 3$ in the diagram

$$\pi_n(\bar{K}^n, \bar{K}^{n-1}, x_0) \xrightarrow{\partial_n} \pi_{n-1}(\bar{K}^{n-1}, x_0) \xrightarrow{j_{n-1}} \pi_{n-1}(\bar{K}^{n-1}, \bar{K}^{n-2}, x_0),$$

and $\gamma_3 = \eta j_2 \partial_3$.

We shall identify the corresponding groups under h. F will always denote the fibre $p^{-1}(x_0)$, and E_n the set $p^{-1}(\bar{K}^n)$. The subscripts $_*$ and $_{**}$ on maps will denote induced homomorphisms of absolute and relative homotopy groups respectively. The subscript $_\sharp$ on a path $\zeta\colon I \to B \subset A$ will denote either of the induced isomorphisms

$$\zeta_\sharp\colon \pi_n(A, B, \zeta(1)) \to \pi_n(A, B, \zeta(0)) \quad \text{or} \quad \zeta_\sharp\colon \pi_n(B, \zeta(1)) \to \pi_n(B, \zeta(0)).$$

If f is a map of $(I^n, \dot{I}^n, I^0)$, $\{f\}$ will mean the homotopy class of f in the relevant group.

† i.e. the isomorphism commutes with the homomorphisms induced by a cellular map $(K, L) \to (K', L')$.

2. The first obstruction for $n \geqslant 1$

Throughout § 2 let g be a cross-section over $\bar{K}^n$ for fixed $n \geqslant 1$, i.e. a map of $\bar{K}^n$ into E such that $pg = 1$. Let $y_0 = g(x_0)$. Then we have homomorphisms

$$\begin{array}{ccccc} \pi_{n+1}(E_{n+1}, E_n, y_0) & \xrightarrow{\partial'} & \pi_n(E_n, y_0) & \xleftarrow{i_*} & \pi_n(F, y_0) \\ \uparrow{\scriptstyle p_{**}^{-1}} & & \uparrow{\scriptstyle g_*} & & \\ \pi_{n+1}(\bar{K}^{n+1}, \bar{K}^n, x_0) & \xrightarrow{\partial} & \pi_n(\bar{K}^n, x_0), & & \end{array} \tag{2.1}$$

where $p \mid E_{n+1}$ induces the isomorphism

$$p_{**}\colon \pi_{n+1}(E_{n+1}, E_n, y_0) \to \pi_{n+1}(\bar{K}^{n+1}, \bar{K}^n, x_0).$$

There exists an exact sequence

$$\ldots \to \pi_q(F, y_0) \xrightarrow{i_*} \pi_q(E_n, y_0) \xrightarrow{p_*} \pi_q(\bar{K}^n, x_0) \xrightarrow{\Delta} \pi_{q-1}(F, y_0) \to \ldots \to \pi_1(\bar{K}^n, x_0).$$

Since $p_* g_* = 1$,

$$\pi_n(E_n, y_0) = i_* \pi_n(F, y_0) + g_* \pi_n(\bar{K}^n, x_0)$$

if $n \geqslant 2$, and $\pi_1(E_1, y_0)$ contains an invariant subgroup which is isomorphic to $\pi_1(F, y_0)$ under i_*, and which is also the kernel of p_* [the proof follows that for fibre bundles; cf. (**4**) 92]. Since†

$$p_*(\partial' p_{**}^{-1} - g_* \partial) = 0,$$

$\partial' p_{**}^{-1} - g_* \partial$ has values in $i_* \pi_n(F, y_0)$; i_* is an isomorphism into, and so we may define

$$c^{n+1}(g) = i_*^{-1}(\partial' p_{**}^{-1} - g_* \partial); \tag{2.2}$$

$c^{n+1}(g)$ is a homomorphism for $n \geqslant 2$, and, for $n = 1$, is a crossed homomorphism (**6**) under the following operation of $\pi_2(\bar{K}^2, \bar{K}^1, x_0)$ on $\pi_1(F, y_0)$: if $\alpha \in \pi_1(F, y_0)$, $\beta \in \pi_2(\bar{K}^2, \bar{K}^1, x_0)$, then

$$\beta(\alpha) = i_*^{-1}(g_* \partial(\beta) + i_*(\alpha) - g_* \partial(\beta)).$$

It is easily verified that this definition gives a group of operators.

[2.3] Definition. *The homomorphism (or crossed homomorphism for $n = 1$) $c^{n+1}(g)$ is called the 'first obstruction' to extending g to a cross-section over $\bar{K}^{n+1}$.*

The superscript will be omitted when it is obvious.

† I shall write all groups additively, even if non-Abelian.

[2.4] THEOREM. *$c^{n+1}(g) = 0$ if and only if g can be extended to a cross-section over $\bar{K}^{n+1}$.*

Proof. Suppose that g can be extended to a cross-section g' over $\bar{K}^{n+1}$. Then $pg' = 1$, and so $p_{**}g'_{**} = 1$, where

$$g'_{**}\colon \pi_{n+1}(\bar{K}^{n+1}, \bar{K}^n, x_0) \to \pi_{n+1}(E_{n+1}, E_n, y_0).$$

Since p_{**} is an isomorphism onto, g'_{**} is its inverse, and

$$\partial' p_{**}^{-1} = \partial' g'_{**} = g_* \partial;$$

thus $c^{n+1}(g) = 0$. For the converse we need

[2.5] LEMMA. *Given any map $\phi\colon S^m \times I \to E_n$ such that*

(i) $p\phi(x, 0) = p\phi(x, 1)$ *for all* $x \in S^m$,

(ii) $p\phi(s, t)$ *is a fixed point* x_1 *for some* $s \in S^m$ *and all* t;

then there exists a map $\phi'\colon S^m \times I \to E_n$ such that

(iii) $\phi'(x, 0) = \phi(x, 0)$, $\phi'(x, 1) = \phi(x, 1)$,

(iv) $p\phi'(x, t) = p\phi'(x, 0)$ *for all* t

(i.e. ϕ' is a fibre-true homotopy of $\phi(\ , 0)$ to $\phi(\ , 1)$).

Proof. Let ζ be a path in $p^{-1}(x_1)$ (which is pathwise connected by § 6, *b*) from $\phi(s \times 0)$ to $gp\phi(s \times 0) = g(x_1)$. Let

$$M = S^m \times 0 \cup S^m \times 1 \cup s \times I.$$

Then $S^m \times I = M \cup e^{m+1}$, where e^{m+1} is an $(m+1)$-cell. We already know the projection $p\phi'$ of the desired map ϕ'. Using difference cochains [cf. (1) § 8 but consider base points], $d(p\phi, p\phi') \in \pi_{m+1}(\bar{K}^n, x_1)$, and thus $\zeta_\sharp g_* d(p\phi, p\phi') \in \pi_{m+1}(E_n, \phi(s \times 0))$. We can therefore construct a map $\phi''\colon S^m \times I \to E_n$ such that $\phi'' \mid M = \phi \mid M$ and $d(\phi, \phi'') = \zeta_\sharp g_* d(p\phi, p\phi')$. From the properties of difference cochains,

$$\begin{aligned} d(p\phi', p\phi'') &= -d(p\phi, p\phi') + d(p\phi, p\phi'') = -d(p\phi, p\phi') + p_* d(\phi, \phi'') \\ &= -d(p\phi, p\phi') + p_* \zeta_\sharp g_* d(p\phi, p\phi') = 0. \end{aligned}$$

Therefore $p\phi' \simeq p\phi''$ rel M. Lift this homotopy to a homotopy rel M of ϕ'' into a map covering $p\phi'$; this last map is the desired ϕ'. This proves the lemma.

Let e^{n+1} be an $(n+1)$-cell of $K - L$; let x_1 be a point on the boundary of e^{n+1}, and set $y_1 = g(x_1)$. Then there exists a path $\xi \subset \bar{K}^n$ from x_1 to x_0. Let Q_i $(i = 0, 1)$ be the category whose objects are the groups

(*a*) $\pi_n(C, D, x_i)$ and $\pi_n(D, x_i)$ for all n, where $K \supset C \supset D \supset \xi(I)$,

(*b*) $\pi_n(C', D', y_i)$ and $\pi_n(D', y_i)$ for all n, where $E \supset C' \supset D' \supset g\xi(I)$,

and whose maps are all homomorphisms of these groups. Then $\xi_\sharp$ and

$(g\xi)_\sharp$ together form a covariant functor from Q_0 to Q_1. In particular, the square in diagram (2.1) is transformed into

$$\begin{array}{ccc} \pi_{n+1}(E_{n+1}, E_n, y_1) & \xrightarrow{\partial'} & \pi_n(E_n, y_1) \\ \uparrow{\scriptstyle p_{**}^{-1}} & & \uparrow{\scriptstyle g_*} \\ \pi_{n+1}(\bar{K}^{n+1}, \bar{K}^n, x_1) & \xrightarrow{\partial} & \pi_n(\bar{K}^n, x_1). \end{array} \tag{2.6}$$

Suppose that $c^{n+1}(g) = 0$; then $g_*\partial = \partial' p_{**}^{-1}$ in (2.1), and the functorial relationship implies that the same equality holds in (2.6). Let

$$f\colon (I^{n+1}, \dot{I}^{n+1}, I^0) \to (\bar{K}^{n+1}, \bar{K}^n, x_1)$$

be a characteristic map for e^{n+1}; f can be lifted to a map

$$\theta\colon (I^{n+1}, \dot{I}^{n+1}, I^0) \to (E_{n+1}, E_n, y_1),$$

and $p_{**}^{-1}\{f\} = \{\theta\} \in \pi_{n+1}(E_{n+1}, E_n, y_1)$. Then $g_*\partial = \partial p_{**}^{-1}$ implies

$$gf \mid \dot{I}^{n+1} \simeq \theta \mid \dot{I}^{n+1} \operatorname{rel} I^0$$

under a homotopy τ in E_n. Therefore, by the lemma, $gf \mid \dot{I}^{n+1} \simeq \theta \mid \dot{I}^{n+1}$ under a homotopy τ', where $p\tau'(x,t) = pgf(x)$ for all x, t. Since

$$I^{n+1} \times 0 \cup \dot{I}^{n+1} \times I$$

is a retract of $I^{n+1} \times I$, there exists a map

$$\theta'\colon I^{n+1} \to E_{n+1}, \qquad \theta' \mid \dot{I}^{n+1} = gf \mid \dot{I}^{n+1},$$

such that $\theta' \simeq \theta$ under a homotopy σ with $\sigma \mid \dot{I}^{n+1} \times I = \tau'$. Thus

$$p\theta' \simeq p\theta \operatorname{rel} \dot{I}^{n+1}.$$

Lifting $p\sigma$ rel $\dot{I}^{n+1}$ to a homotopy of θ', we obtain a map $\theta''\colon I^{n+1} \to E_{n+1}$ which covers $p\theta = f$. Now $\theta'' \mid \dot{I}^{n+1} = gf \mid \dot{I}^{n+1}$; thus $\theta'' f^{-1}$ is single-valued. Since $\bar{e}^{n+1}$ has the identification topology under f, $\theta'' f^{-1}$ is continuous.

Let $g'\colon \bar{K}^{n+1} \to E$ be defined by $g' \mid \bar{K}^n = g$ and $g' \mid \bar{e}^{n+1} = \theta'' f^{-1}$ for each $e^{n+1} \subset \bar{K}^{n+1} - L$. Then g' is single-valued and continuous, since $\bar{K}^{n+1}$ has the weak topology and is a cross-section extending g.

3. Operators

Let $y_0 \in F$ be a base point, and suppose that there exists a cross-section g_1 over K^1 which is extendible to K^2, and such that $g_1(x_0) = y_0$ (e.g. this condition holds for any $y_0 \in F$ if F is 1-connected). $\pi_1(K, x_0)$ acts as a group of operators on $\pi_m(E_n, y_0)$ for $n \geqslant 2$, $m \geqslant 1$ as follows: any $\xi \in \pi_1(K, x_0)$ has a representative map $f\colon (I, \dot{I}) \to (K^1, x_0)$. Then $g_1 f$ represents an element $\zeta \in \pi_1(E_n, y_0)$, and the operation of ξ on

$$\beta \in \pi_m(E_n, y_0)$$

is defined to be $\xi_{E_n}(\beta) = \zeta(\beta)$.

$\pi_1(K, x_0)$ also operates on $\pi_m(F, y_0)$ through g_1 as follows: let

$$\alpha \in \pi_m(F, y_0)$$

have representative k: $(I^m, \dot{I}^m) \to (F, y_0)$ and let ξ have representative f as before. Define a map H: $I^m \times I \to K^1$ by $H(x,t) = f(1-t)$, and a map $\tilde{H}'$: $\dot{I}^m \times I \to E_1$ by $\tilde{H}'(y,t) = g_1 f(1-t)$. Then by [1.1] there exists a homotopy $\tilde{H}$ of k which extends $\tilde{H}'$ and such that $p\tilde{H} = H$. The operation of ξ on α is defined to be $\xi_F(\alpha) = \{\tilde{H}(\ ,1)\} \in \pi_m(F, y_0)$. The extendibility of g_1 shows that the operation is independent of the representative maps. Under this definition $\pi_1(K, x_0)$ is a group of operators, and we have the relation

$$i_* \xi_F(\alpha) = \xi_{E_n}(i_* \alpha) \quad \text{for } \alpha \in \pi_n(F, y_0). \tag{3.1}$$

Different cross-sections g_1 may give different operations.

With certain mild restrictions, $\pi_1(K, x_0)$ also operates on $\pi_m(F, y_0)$ through the group of homotopy equivalences $(F, y_0) \to (F, y_0)$: e.g. let F be a CW-complex, y_0 a 0-cell, and g_1 a cross-section as above. Let ξ and f be as above, and define H: $F \times I \to K^1$ by $H(y,t) = f(1-t)$, $y \in F$. Then, if i: $F \subset E$ is the inclusion, H is a homotopy of pi. By [1.1], H can be lifted to a homotopy $\tilde{H}$ of i such that $\tilde{H}(y_0, t) = g_1 f(1-t)$. $\tilde{H}(\ ,1)$ is a homotopy equivalence of (F, y_0), and thus induces an automorphism of $\pi_m(F, y_0)$ which is defined to be the operation of ξ. This operation is easily seen to coincide with the one previously defined.

$C_r(\bar{K}, \bar{L})$ also admits $\pi_1(K, x_0)$ as a group of operators, so that we may define $C^r(\bar{K}, \bar{L}; \pi_m(F, y_0)) = \text{ophom}(C_r(\bar{K}, \bar{L}), \pi_m(F, y_0))$, the *equivariant* cochain group [cf. (**8**) § 2].

4. The obstruction class for $n \geqslant 2$

Take $n \geqslant 2$ throughout this section, and let g be a cross-section over $\bar{K}^n$. Set $y_0 = g(x_0)$, and let $\pi_1(K, x_0)$ operate on $\pi_n(F, y_0)$ as defined in § 3, with $g_1 = g \mid K^1$.

[4.1] THEOREM. *If g' is a cross-section over $\bar{K}^n$ such that*

$$g' \mid \bar{K}^{n-1} = g \mid \bar{K}^{n-1},$$

then (a) $c^{n+1}(g')$ is a cocycle in the equivariant cochain group

$$C^{n+1}(\bar{K}, \bar{L}; \pi_n(F, y_0)),$$

and (b) $c^{n+1}(g')$ and $c^{n+1}(g)$ differ only by a coboundary.

(c) If α is cohomologous to $c^{n+1}(g)$, then there exists a cross-section g'' over $\bar{K}^n$,

$$g'' \mid \bar{K}^{n-1} = g \mid \bar{K}^{n-1},$$

such that $\alpha = c^{n+1}(g'')$.

[4.2] DEFINITION. *For $n \geqslant 2$, we shall call the cohomology class of $c^{n+1}(g)$ in the equivariant cohomology group $H^{n+1}(\bar{K}, \bar{L}; \pi_n(F, y_0))$ the 'obstruction class'† of g, written $\bar{c}^{n+1}(g)$.*

[4.3] COROLLARY. *$\bar{c}^{n+1}(g) = 0$ if and only if $g \mid \bar{K}^{n-1}$ can be extended to a cross-section over $\bar{K}^{n+1}$; i.e. if and only if g can be re-defined on the n-cells of $K - L$ so as to be extendible to $K^{n+1} \cup L$.*

Proof of [4.1]. (*a*) Since $g' \mid K^1 = g \mid K^1$, $c^{n+1}(g')$ is an operator homomorphism. Consider

$$\begin{array}{ccccccc}
\pi_{n+2}(E_{n+2}, E_{n+1}, y_0) & \xrightarrow{\gamma'_{n+2}} & \pi_{n+1}(E_{n+1}, E_n, y_0) & \xrightarrow{\partial'_{n+1}} & \pi_n(E_n, y_0) & \xleftarrow{i_*} & \pi_n(F, y_0) \\
\uparrow p_{**}^{-1} & & \uparrow p_{**}^{-1} & & \uparrow g'_* & & \\
\pi_{n+2}(\bar{K}^{n+2}, \bar{K}^{n+1}, x_0) & \xrightarrow{\gamma_{n+2}} & \pi_{n+1}(\bar{K}^{n+1}, \bar{K}^n, x_0) & \xrightarrow{\partial_{n+1}} & \pi_n(\bar{K}^n, x_0). & &
\end{array}$$

The first square commutes. Therefore

$$c(g')\gamma_{n+2} = i_*^{-1}(\partial'_{n+1} j'_{n+1} \partial'_{n+2} p_{**}^{-1} - g'_* \partial_{n+1} j_{n+1} \partial_{n+2}) = 0,$$

since $\partial_{n+1} j_{n+1} = 0 = \partial'_{n+1} j'_{n+1}$. Thus, if δ_q is the coboundary operator in the equivariant cochain system $C^q(\bar{K}, \bar{L}; \pi_n(F, y_0))$,

$$\delta_{n+1} c(g') = c(g')\gamma_{n+2} = 0,$$

and $c(g')$ is a cocycle.

(*b*) The separation homomorphism

$$\Delta(g, g') : \pi_n(\bar{K}^n, \bar{K}^{n-1}, x_0) \to \pi_n(E_n, y_0)$$

is defined as in (7) Appendix B, but with I^0 as base point in I^n; clearly the relation $g_* - g'_* = \Delta j$ still holds. From the convention for orienting the hemispheres it follows that $p_* \Delta = 0$, so Δ maps into $i_* \pi_n(F, y_0)$. [3.1] may be used to show that $\Delta' = i_*^{-1} \Delta$ is an operator homomorphism; therefore, for $n \geqslant 3$,

$$c(g') - c(g) = \Delta' j \partial = \Delta' \gamma = \delta \Delta'.$$

If $n = 2$, then Δ' maps into an Abelian group and can be factored through the projection η, $\Delta' = \Delta'' \eta$. Δ'' is an operator homomorphism, so that

$$c(g') - c(g) = \Delta'' \eta j \partial = \Delta'' \gamma = \delta \Delta''.$$

(*c*) We wish to show that, given $k \in C^n(\bar{K}, \bar{L}; \pi_n(F, y_0))$, there exists a cross-section g'' over $\bar{K}^n$, agreeing with g on $\bar{K}^{n-1}$, such that $\Delta'(g, g'') = k$

† Alternatively, by (4.3), $\bar{c}^{n+1}(g)$ is the second obstruction to $g \mid \bar{K}^{n-1}$.

if $n \geqslant 3$, and $\Delta''(g, g') = k$ if $n = 2$. For, by hypothesis, $\alpha - c(g) = \delta k$ for some such k. Then, if $n \geqslant 3$,

$$\alpha - c(g) = \delta\Delta' = c(g'') - c(g),$$

while, if $n = 2$, $\quad \alpha - c(g) = \delta\Delta'' = c(g'') - c(g)$;

and in both cases $c(g'') = \alpha$.

First consider the case $n \geqslant 3$. Let $(e_\iota^n)_{\iota \in A}$ be the set of n-cells in $K - L$, and for each $\iota \in A$ let f_ι be a characteristic map for e_ι^n. Set $z_\iota = f_\iota(I^0)$, $w_\iota = g(z_\iota)$, and let ζ_ι be a path in $\bar{K}^{n-1}$ from x_0 to z_ι. f_ι represents an element $\{f_\iota\} \in \pi_n(\bar{K}^n, \bar{K}^{n-1}, z_\iota)$, and the set $(\zeta_{\iota *}\{f_\iota\})_{\iota \in A}$ is a basis for the free $\pi_1(K, x_0)$-module $C_n(\bar{K}, \bar{L})$. Set $\rho_\iota = \zeta_{\iota *}\{f_\iota\}$.

Let g'' be a cross-section over $\bar{K}^n$ which agrees with g on $\bar{K}^{n-1}$. Then

$$\Delta(g, g'')(\rho_\iota) = (g\zeta_\iota)_* \Delta_\iota(g, g'')\{f_\iota\}, \tag{4.4}$$

where Δ_ι is the separation homomorphism for base point z_ι. Furthermore, $\Delta(g, g'')(\rho_\iota)$ depends only on those values of g and g'' on the point set $\bar{e}_\iota^n$. Thus we need only show that, given the element $k(\rho_\iota) \in \pi_n(F, y_0)$, we can construct a cross-section g'' over $\bar{K}^n$, agreeing with g on $\bar{K}^n - e_\iota^n$, such that

$$\Delta_\iota(g, g'')\{f_\iota\} = (g\zeta_\iota)_*^{-1} i_* k(\rho_\iota) = \beta_\iota \text{ (say)}.$$

Note that $\beta_\iota \in i_{\iota *}\pi_n(F_\iota, w_\iota)$, where $F_\iota = p^{-1}(z_\iota)$ and $i_\iota \colon F_\iota \subset E_n$.

Let s be a base point on the equator of S^n [cf. (7), Appendix B for orientation]. Let σ map S^n onto E_+^n by $\sigma(t_0, \ldots, t_n) = (t_0, \ldots, -t_n)$ for $(t_0, \ldots, t_n) \in E_-^n$, otherwise by the identity; and let $\tau \colon (E_+^n, s) \to (I^n, I^0)$ be an orientation-preserving homeomorphism. Let β_ι be represented by a map $b \colon (S^n, s) \to (F_\iota, w_\iota)$. Let $\tilde{H}_1$ be a homotopy rel s in E_n of $gf\tau$ into w_ι, $\tilde{H}_2$ a homotopy rel s in F_ι of $b \mid E_+^n$ into w_ι. Let $\tilde{H}'$ be the homotopy $\tilde{H}_2$ followed by the reverse of $\tilde{H}_1$; it is thus a map $\tilde{H}' \colon E_+^n \times I \to E_n$ with $\tilde{H}'(\ ,0) = b \mid E_+^n$, $\tilde{H}'(\ ,1) = gf\tau = gf\tau\sigma \mid E_+^n$. Set

$$H = p\tilde{H}'(\sigma \times 1) \colon S^n \times I \to \bar{K}^n.$$

We can then apply [1.1] with $R = S^n$, $S = E_+^n$, and $\theta = b$, to obtain a homotopy $\tilde{H}$ of b which covers H and agrees with $\tilde{H}'$ on E_+^n. Define $b' \colon I^n \to E_n$ by $b' = (\tilde{H}(\ ,1) \mid E_-^n)(\sigma \mid E_-^n)^{-1}\tau^{-1}$; then b' covers f and agrees with gf on $\dot{I}^n$. Set $g'' = b'f^{-1}$ on e_ι^n, $g'' = g$ otherwise; g'' is a cross-section [cf. the end of § 2] with the desired properties.

Now suppose $n = 2$. Using the notation above, we see that $(\eta\rho_\iota)_{\iota \in A}$ is a basis for the free $\pi_1(K, x_0)$-module $C_2(\bar{K}, \bar{L})$. The last two paragraphs apply, except for replacing k by $k\eta$, to give a cross-section g'' such that $\Delta(g, g'')(\rho_\iota) = i_* k\eta(\rho_\iota)$ for all $\iota \in A$. Thus $\Delta''(\eta\rho_\iota) = k(\eta\rho_\iota)$ for all $\iota \in A$, and $\Delta'' = k$.

5. The obstruction class for $n = 1$

In this section let L be a non-empty connected subcomplex, with $x_0 \in L$, and let g be a cross-section over $\bar{K}^1$. Then $\pi_1(\bar{K}^1, x_0)$ operates on $\pi_1(E_1, y_0)$ through g; i.e. if $\xi \in \pi_1(\bar{K}^1, x_0)$, $\alpha \in \pi_1(E_1, y_0)$, then

$$\xi_{E_1}(\alpha) = g_*(\xi) + \alpha - g_*(\xi).$$

$\pi_1(\bar{K}^1, x_0)$ is also a group of operators on $\pi_1(F, y_0)$ defined by

$$\xi_F(\beta) = i_*^{-1}(\xi_{E_1}(i_*\beta)).$$

The term 'crossed homomorphism' will imply that $\pi_1(\bar{K}^1, x_0)$ acts in this manner.

[5.1] THEOREM. *Let $m_*: \pi_1(L, x_0) \to \pi_1(\bar{K}^1, x_0)$ be the injection. Then $g \mid \bar{K}^0$ can be extended to a cross-section over $\bar{K}^2$ if and only if $c^2(g) = k\partial$, where $k: \pi_1(\bar{K}^1, x_0) \to \pi_1(F, y_0)$ is any crossed homomorphism such that $km_* = 0$.*

Proof. We first establish some preliminaries.

If g' is any cross-section over $\bar{K}^1$ with $g'(x_0) = y_0$, then $i_*^{-1}(g'_* - g_*)$ is a crossed homomorphism, and

$$-c(g') + c(g) = i_*^{-1}(g'_* - g_*)\partial. \tag{5.2}$$

Suppose given a crossed homomorphism $k: \pi_1(\bar{K}^1, x_0) \to \pi_1(F, y_0)$ such that $km_* = 0$; we wish to construct a cross-section g'' over $\bar{K}^1$,

$$g'' \mid \bar{K}^0 = g \mid \bar{K}^0$$

such that $k = i_*^{-1}(g''_* - g_*)$. First choose a fixed characteristic map for each 1-cell. Let T be a tree in L^1 containing all the vertices of L, and extend it to a tree $S \subset K^1$ containing all the vertices of K, and such that $S \cap L = T$ [cf. (9) 322]. Let $(e_\iota)_{\iota \in B}$ be the set of 1-cells in $K - S$. Let e_ι have initial and terminal points z_ι and v_ι (as determined by the characteristic map f_ι) and let ζ_ι, μ_ι be the paths in S (unique to homotopy in S) from x_0 to z_ι and v_ι respectively. Let $\rho_\iota = \{\zeta_\iota + f_\iota - \mu_\iota\} \in \pi_1(\bar{K}^1, x_0)$; the set $(\rho_\iota)_{\iota \in B}$ generates $\pi_1(\bar{K}^1, x_0)$.

Let $F_\iota = p^{-1}(z_\iota)$, $w_\iota = g(z_\iota)$, $i_\iota: F_\iota \subset E$. Let $(e_\iota)_{\iota \in A}$ be the set of 1-cells in $K - (L \cup S)$. Then, given $\iota \in A$ and any element $\alpha_\iota \in i_{\iota *}\pi_1(F_\iota, w_\iota)$, by the argument of the penultimate paragraph of § 4 we can find a cross-section g''_ι over e_ι, agreeing with g on $\dot{e}_\iota$, such that† $\{gf_\iota - g''_\iota f_\iota\}_\iota = \alpha_\iota$. Define $g'' \mid L \cup S = g \mid L \cup S$, and define g'' on each e_ι, $\iota \in A$, such that

$$\{gf_\iota - g''f_\iota\}_\iota = -(g\zeta_\iota)_\sharp^{-1} i_* k(\rho_\iota) \in i_{\iota *}\pi_1(F_\iota, w_\iota). \tag{5.3}$$

Then

$$\begin{aligned}(g_* - g''_*)(\rho_\iota) &= \{g\zeta_\iota + gf_\iota - g\mu_\iota + g''\mu_\iota - g''f_\iota - g''\zeta_\iota\}_0 \\ &= (g\zeta_\iota)_\sharp\{gf_\iota - g''f_\iota\}_\iota = -i_* k(\rho_\iota).\end{aligned}$$

† A subscript after brackets indicates the base point.

Since this equality is obvious for $\iota \in B-A$,

$$k(\rho_\iota) = i_*^{-1}(g''_*-g_*)(\rho_\iota) \quad \text{for all } \iota \in B. \tag{5.4}$$

Since a crossed homomorphism is uniquely determined by its values on a set of generators, $k = i_*^{-1}(g''_*-g_*)$ as desired.

To prove the theorem, first suppose that there exists a cross-section g' over $\bar{K}^2$ with $g' \mid \bar{K}^0 = g \mid \bar{K}^0$. Then $c(g') = 0$, and (5.2) becomes

$$c(g) = i_*^{-1}(g'_*-g_*)\partial.$$

Since $g' \mid L = g \mid L$, $i_*^{-1}(g'_*-g_*)m_* = 0$.

Conversely, suppose that $c(g) = k\partial$, with k as in [5.1]. Then there exists a cross-section g' over $\bar{K}^1$, $g' \mid \bar{K}^0 = g \mid \bar{K}^0$, such that

$$i_*^{-1}(g'_*-g_*) = k.$$

Then $-c(g')+c(g) = k\partial$, so that $c(g') = 0$, and g' can be extended to $\bar{K}^2$.

The case of L empty is equivalent to $L = x_0$ since g is never varied on x_0.

6. Lower dimensions

Let g be a cross-section over $\bar{K}^0$.

[6.1] THEOREM. *We can extend g to a cross-section over $\bar{K}^1$ if and only if $\pi_0(F) = 0$. We can extend $g \mid L$ to a cross-section over $\bar{K}^1$ if and only if $\pi_0(F) = 0$.*

Proof. The theorem is a consequence of the following two observations:

(*a*) $\pi_0(F) = 0$ implies that g can be extended to $\bar{K}^1$. For, let e be any 1-cell in $K-L$, with characteristic map f. Since K and F are pathwise connected, so is $p^{-1}f(0)$. Therefore, if we lift f to a map θ such that $\theta(1) = gf(1)$, then $\theta \mid \dot{I} \simeq gf \mid \dot{I}$ in a fibre-true homotopy, and the method of § 2 (much simplified) gives an extension of g to a cross-section over e.

(*b*) If there exists a cross-section g' over K^1, then $\pi_0(F) = 0$. For, let $y_0 = g'(x_0)$ and let w be any point of F. Then there exists a path μ in E from y_0 to w. Let ζ be a path in K^1 from x_0 to x_0 such that $p\mu \simeq \zeta \operatorname{rel} \dot{I}$. Lift the homotopy rel $\dot{I}$ to obtain a path ζ' from y_0 to w which covers ζ. Then $-\zeta'+g'\zeta$ covers $-\zeta+\zeta$, and, if we shrink the latter path to x_0 rel $\dot{I}$, and lift the homotopy rel $\dot{I}$, we obtain a path in F from w to y_0.

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ON THE SECOND THEOREM OF CONSISTENCY IN THE THEORY OF ABSOLUTE SUMMABILITY

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1.1. *Definitions.* Let $\sum c_n$ be a given infinite series, and λ_n a positive, steadily increasing, monotonic function of n, tending to infinity with n.

We write
$$C_\lambda(\omega) = C_\lambda^0(\omega) = \sum_{\lambda_n \leqslant \omega} c_n,$$

and
$$C_\lambda^r(\omega) = \sum_{\lambda_n \leqslant \omega} (\omega - \lambda_n)^r c_n = r \int_{\lambda_1}^{\omega} C_\lambda(\tau)(\omega-\tau)^{r-1}\, d\tau \quad (r > 0).$$

The series $\sum c_n$ is said to be *summable* (R, λ, r) $(r \geqslant 0)$ to C if
$$C_\lambda^r(\omega)/\omega^r \to C,$$
as $\omega \to \infty$ [(**8**)].

The series $\sum c_n$ is said to be *absolutely summable* (R, λ, r), or *summable* $|R, \lambda, r|$ $(r \geqslant 0)$ if
$$C_\lambda^r(\omega)/\omega^r$$
is of bounded variation in (h, ∞), where h is a finite positive number [(**6**) and (**7**)].

1.2. Extending the well-known 'second theorem of consistency' for Riesz summability [(**3**) 30–32], Hardy (**2**) proved in 1916 the following theorem.

THEOREM A. *If the series $\sum c_n$ is summable (R, λ, κ) to sum C, and μ is a logarithmico-exponential function of λ such that*
$$\mu = O(\lambda^\Delta),$$
where Δ is a constant, then the series $\sum c_n$ is summable (R, μ, κ) to the same sum C.

In 1932 Hirst (**4**) gave a generalization of Hardy's theorem by replacing μ by a more general function. Very recently Kuttner (**5**) has shown that under certain restriction Hirst's conditions become necessary and sufficient.

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Working on the lines of Hardy's theorem, Chandrasekharan (**1**) gave an analogous theorem for the absolute summability of series by Riesz means of the type μ, which is a special class of logarithmico-exponential function. The object of the present paper is to extend the scope of applicability of the second theorem of consistency for absolutely summable series, and to establish a theorem applicable to a wider class of types.

I should like to acknowledge my indebtedness to Dr. B. N. Prasad for his valuable guidance during the preparation of this paper.

2.1. I establish the following theorem.

THEOREM B. *If $\phi(t)$ is a non-negative and monotonic increasing function of t for $t \geqslant 0$, steadily tending to infinity as t tends to infinity, such that, for positive integral κ, $\phi(t)$ is a $(\kappa+1)$th indefinite integral for $t \geqslant 0$, and*

$$t^r\phi^{(r)}(t)/\phi(t) \in \mathrm{BV}(h, \infty) \quad (r = 1, 2, \ldots, \kappa), \dagger \tag{2.11}$$

where h is a finite positive number, then any infinite series which is summable $|R, \lambda_n, \kappa|$ is also summable $|R, \phi(\lambda_n), \kappa|$.

It is evident that the truth or otherwise of the theorem depends only upon the behaviour of $\phi(t)$ for sufficiently large t. We may therefore alter $\phi(t)$ in any finite range in any convenient way, and may suppose, without any loss of generality, that $h = \lambda_1$, and that $\phi(t)$ is a $(\kappa+1)$th indefinite integral for $t \geqslant 0$ (instead of only for sufficiently large t).

2.2. We require the following lemmas for the proof of the theorem.

LEMMA 1.‡ *If κ is a positive integer, then*

$$C_\lambda(\sigma) = \frac{1}{\kappa!}\left(\frac{d}{d\sigma}\right)^\kappa C_\lambda^\kappa(\sigma).$$

LEMMA 2. *The n-th derivative of $\{f(x)\}^m$ is the sum of a number of terms of the form*

$$K\{f(x)\}^{m-r}\{f^{(1)}(x)\}^{\alpha_1}\{f^{(2)}(x)\}^{\alpha_2}\ldots\{f^{(n)}(x)\}^{\alpha_n},$$

where K is a constant,§ $r \leqslant n$, and the α's are positive integers or zeros such that

$$\sum_{\nu=1}^{n} \alpha_\nu = r, \qquad \sum_{\nu=1}^{n} \nu\alpha_\nu = n.$$

Further, if m is a positive integer, then $0 < r \leqslant m$.

† By '$f(x) \in \mathrm{BV}(h, k)$' we mean that $f(x)$ is a function of bounded variation in the interval (h, k).

‡ Hardy and Riesz (**3**).

§ Throughout, K denotes a constant, not necessarily the same at each occurrence.

This is a particular case of a result, due to Faa di Bruno [(**9**) 89] on the nth derivative of a function of a function; the factor $\{f(x)\}^{m-r}$ accrues from the differentiation of $\{f(x)\}^m$ with respect to $f(x)$, and is multiplied by a zero factor if m is a positive integer and $r > m$.

LEMMA 3. *Let δ be a finite positive number, and let $\phi(t)$ be a non-negative and monotonic increasing function of t for $t \geqslant 0$. If*

$$G(\sigma) \in \mathrm{BV}(\delta, \infty)$$

and

$$\frac{1}{\{\phi(\eta)\}^r} \int_\delta^\eta H(\sigma)\, d\sigma \in \mathrm{BV}(\delta, \infty) \quad (r > 0)$$

then

$$\frac{1}{\{\phi(\eta)\}^r} \int_\delta^\eta H(\sigma) G(\sigma)\, d\sigma \in \mathrm{BV}(\delta, \infty).$$

Proof. Integrating by parts, we have

$$\frac{1}{\{\phi(\eta)\}^r} \int_\delta^\eta H(\sigma) G(\sigma)\, d\sigma = \frac{G(\eta)}{\{\phi(\eta)\}^r} \int_\delta^\eta H(\sigma)\, d\sigma -$$

$$- \frac{1}{\{\phi(\eta)\}^r} \int_\delta^\eta G^{(1)}(\sigma) \left\{ \int_\delta^\sigma H(\tau)\, d\tau \right\} d\sigma.$$

The first term is clearly of bounded variation in (δ, ∞). The derivative of the second term is $I_1(\eta) + I_2(\eta)$, where

$$I_1(\eta) = - \frac{G^{(1)}(\eta)}{\{\phi(\eta)\}^r} \int_\delta^\eta H(\sigma)\, d\sigma,$$

$$I_2(\eta) = \frac{r\phi^{(1)}(\eta)}{\{\phi(\eta)\}^{r+1}} \int_\delta^\eta G^{(1)}(\sigma) \left\{ \int_\delta^\sigma H(\tau)\, d\tau \right\} d\sigma.$$

Now $I_1(\eta)$ is the product of $G^{(1)}(\eta)$ and a function which is of bounded variation, and hence bounded, in the interval (δ, ∞). Therefore

$$\int_\delta^\infty |I_1(\eta)|\, d\eta < \infty.$$

Also

$$\int_\delta^X |I_2(\eta)|\,d\eta \leqslant \int_\delta^X \frac{r\phi^{(1)}(\eta)}{\{\phi(\eta)\}^{r+1}}\left(\int_\delta^\eta |G^{(1)}(\sigma)|\left|\int_\delta^\sigma H(\tau)\,d\tau\right|d\sigma\right)d\eta$$

$$= \int_\delta^X |G^{(1)}(\sigma)|\left|\int_\delta^\sigma H(\tau)\,d\tau\right|\left\{\int_\sigma^X \frac{r\phi^{(1)}(\eta)\,d\eta}{\{\phi(\eta)\}^{r+1}}\right\}d\sigma$$

$$= \int_\delta^X |G^{(1)}(\sigma)|\frac{1}{\{\phi(\sigma)\}^r}\left|\int_\delta^\sigma H(\tau)\,d\tau\right|\left\{1-\frac{\{\phi(\sigma)\}^r}{\{\phi(X)\}^r}\right\}d\sigma$$

$$\leqslant K\int_\delta^X |G^{(1)}(\sigma)|\,d\sigma,$$

since
$$\int_\delta^\sigma H(\tau)\,d\tau = O(\{\phi(\sigma)\}^r),$$

and
$$0 \leqslant 1-\frac{\{\phi(\sigma)\}^r}{\{\phi(X)\}^r} \leqslant 1.$$

Hence the result.

LEMMA 4. *If*

$$\chi(\eta) = \int_{\lambda_1}^\eta \sigma^\kappa\{\phi(\sigma)\}^\alpha\{\phi^{(1)}(\sigma)\}^{\alpha_1}\ldots\{\phi^{(\kappa+1)}(\sigma)\}^{\alpha_{\kappa+1}}\,d\sigma,$$

where the α's are positive integers or zeros such that

$$0 < \alpha+\alpha_1+\alpha_2+\ldots+\alpha_{\kappa+1} = r \leqslant \kappa$$

and
$$\alpha_1+2\alpha_2+\ldots+(\kappa+1)\alpha_{\kappa+1} = \kappa+1,$$

then, under the hypotheses (2.11) *and* $h = \lambda_1$,

$$\chi(\eta)/\{\phi(\eta)\}^r \in \mathrm{BV}(\lambda_1,\infty).$$

Proof. Case (i): $\alpha_{\kappa+1} > 0$.

In this case evidently

$$\alpha_1 = \alpha_2 = \ldots = \alpha_\kappa = 0,$$

and $\alpha = r-1$. Hence

$$\frac{\chi(\eta)}{\{\phi(\eta)\}^r} = \frac{1}{\{\phi(\eta)\}^r}\int_{\lambda_1}^\eta \sigma^\kappa\{\phi(\sigma)\}^{r-1}\phi^{(\kappa+1)}(\sigma)\,d\sigma.$$

We seek to prove that, for $r > 0$, $n = 0, 1, 2, \ldots, \kappa$,

$$(2.21) \qquad F_n(\eta) = \frac{1}{\{\phi(\eta)\}^r} \int_{\lambda_1}^{\eta} \sigma^n \{\phi(\sigma)\}^{r-1} \phi^{(n+1)}(\sigma)\, d\sigma \in \mathrm{BV}(\lambda_1, \infty);$$

the result required for the proof of the present case of the lemma is the special one in which $n = \kappa$. The result (2.21) evidently holds for $n = 0$. So, it suffices to assume it true for n $(0 \leqslant n \leqslant \kappa - 1)$ and prove it true for $n+1$.

Integrating by parts, we get

$$F_{n+1}(\eta) = \frac{\eta^{n+1}\phi^{(n+1)}(\eta)}{\phi(\eta)} - \frac{K}{\{\phi(\eta)\}^r} - \frac{(n+1)}{\{\phi(\eta)\}^r} \int_{\lambda_1}^{\eta} \sigma^n \{\phi(\sigma)\}^{r-1} \phi^{(n+1)}(\sigma)\, d\sigma -$$

$$- \frac{(r-1)}{\{\phi(\eta)\}^r} \int_{\lambda_1}^{\eta} \sigma^{n+1} \{\phi(\sigma)\}^{r-2} \phi^{(1)}(\sigma) \phi^{(n+1)}(\sigma)\, d\sigma.$$

The first term is of bounded variation in (λ_1, ∞) by hypothesis. The second term is evidently of bounded variation in (λ_1, ∞). The third term is of bounded variation in (λ_1, ∞) by the induction-hypothesis. The last term may be written as

$$- \frac{(r-1)}{\{\phi(\eta)\}^r} \int_{\lambda_1}^{\eta} \sigma^n \{\phi(\sigma)\}^{r-1} \phi^{(n+1)}(\sigma) \frac{\sigma\phi^{(1)}(\sigma)}{\phi(\sigma)}\, d\sigma,$$

which is of bounded variation in (λ_1, ∞) by the induction-hypothesis and Lemma 3, since

$$\frac{\sigma\phi^{(1)}(\sigma)}{\phi(\sigma)} \in \mathrm{BV}(\lambda_1, \infty).$$

Case (ii): $\alpha_{\kappa+1} = 0$.

In this case $\qquad \alpha_1 + 2\alpha_2 + \ldots + \kappa\alpha_\kappa = \kappa + 1$

and $\qquad \alpha + \alpha_1 + \alpha_2 + \ldots + \alpha_\kappa = r \leqslant \kappa.$

Since $\qquad \alpha_1 + 2\alpha_2 + \ldots + \kappa\alpha_\kappa = \kappa + 1,$

there exists at least one index, say α_n, which is not zero. Hence

$$\chi(\eta) = \int_{\lambda_1}^{\eta} \sigma^{\kappa}\{\phi(\sigma)\}^{r-\alpha_1-\alpha_2-\ldots-\alpha_\kappa}\{\phi^{(1)}(\sigma)\}^{\alpha_1}\{\phi^{(2)}(\sigma)\}^{\alpha_2}\ldots\{\phi^{(\kappa)}(\sigma)\}^{\alpha_\kappa}\,d\sigma$$

$$= \int_{\lambda_1}^{\eta} \sigma^{n-1}\{\phi(\sigma)\}^{r-1}\phi^{(n)}(\sigma)\left(\frac{\sigma\phi^{(1)}(\sigma)}{\phi(\sigma)}\right)^{\alpha_1}\ldots\left(\frac{\sigma^n\phi^{(n)}(\sigma)}{\phi(\sigma)}\right)^{\alpha_n-1}\ldots\left(\frac{\sigma^\kappa\phi^{(\kappa)}(\sigma)}{\phi(\sigma)}\right)^{\alpha_\kappa} d\sigma$$

$$= \int_{\lambda_1}^{\eta} \sigma^{n-1}\{\phi(\sigma)\}^{r-1}\phi^{(n)}(\sigma)G(\sigma)\,d\sigma, \text{ say.}$$

Thus $$\chi(\eta)/\{\phi(\eta)\}^r = \frac{1}{\{\phi(\eta)\}^r}\int_{\lambda_1}^{\eta} \sigma^{n-1}\{\phi(\sigma)\}^{r-1}\phi^{(n)}(\sigma)G(\sigma)\,d\sigma,$$

where $$G(\sigma) \in \mathrm{BV}(\lambda_1, \infty).$$

It has already been shown that, for $n = 1, 2, \ldots, \kappa$,

$$\frac{1}{\{\phi(\eta)\}^r}\int_{\lambda_1}^{\eta} \sigma^{n-1}\{\phi(\sigma)\}^{r-1}\phi^{(n)}(\sigma)\,d\sigma \in \mathrm{BV}(\lambda_1, \infty).$$

Hence, applying Lemma 3, we obtain the result.

2.3. *Proof of the theorem.* By hypothesis,

$$R_\lambda^\kappa(\eta) = C_\lambda^\kappa(\eta)/\eta^\kappa = \frac{\kappa}{\eta^\kappa}\int_{\lambda_1}^{\eta} C_\lambda(\sigma)(\eta-\sigma)^{\kappa-1}\,d\sigma \in \mathrm{BV}(\lambda_1, \infty).$$

We have to show that

$$(2.31) \qquad K_\mu^\kappa(\zeta) = -\frac{1}{\zeta^\kappa}\int_{\lambda_1}^{\eta} C_\lambda(\sigma)\frac{d}{d\sigma}\{\phi(\eta)-\phi(\sigma)\}^\kappa\,d\sigma \in \mathrm{BV}(\lambda_1, \infty),$$

where $$\mu = \phi(\lambda), \qquad \zeta = \phi(\eta).$$

By Lemma 1, $R_\mu^\kappa(\zeta)$ is a constant multiple of

$$\frac{1}{\{\phi(\eta)\}^\kappa}\int_{\lambda_1}^{\eta}\left(\frac{d}{d\sigma}\right)^\kappa C_\lambda^\kappa(\sigma)\frac{d}{d\sigma}\{\phi(\eta)-\phi(\sigma)\}^\kappa\,d\sigma.$$

Since $C_\lambda^\kappa(\sigma)$ and its first $\kappa-1$ derivatives vanish for $\sigma = \lambda_1$, the integral in the above expression equals

$$(-1)^{\kappa-1}C_\lambda^\kappa(\eta)\left[\left(\frac{d}{d\sigma}\right)^\kappa\{\phi(\eta)-\phi(\sigma)\}^\kappa\right]_{\sigma=\eta} + \\ +(-1)^\kappa\int_{\lambda_1}^{\eta} C_\lambda^\kappa(\sigma)\left(\frac{d}{d\sigma}\right)^{\kappa+1}\{\phi(\eta)-\phi(\sigma)\}^\kappa\,d\sigma$$

$$= -\kappa!\,C_\lambda^\kappa(\eta)\{\phi^{(1)}(\eta)\}^\kappa+(-1)^\kappa\int_{\lambda_1}^{\eta} C_\lambda^\kappa(\sigma)\left(\frac{d}{d\sigma}\right)^{\kappa+1}\{\phi(\eta)-\phi(\sigma)\}^\kappa\,d\sigma$$

$$= \mathscr{F}_1+\mathscr{F}_2, \text{ say.}$$

Now
$$\mathscr{F}_1/\{\phi(\eta)\}^\kappa = -\kappa!\,\frac{C_\lambda^\kappa(\eta)}{\eta^\kappa}\left\{\frac{\eta\phi^{(1)}(\eta)}{\phi(\eta)}\right\}^\kappa.$$

Thus, by hypothesis, $\mathscr{F}_1/\{\phi(\eta)\}^\kappa$ is the product of two functions each of which is of bounded variation in (λ_1,∞), and, therefore,

$$\mathscr{F}_1/\{\phi(\eta)\}^\kappa \in \mathrm{BV}(\lambda_1,\infty).$$

Now, we have only to show that

$$\mathscr{F}_2/\{\phi(\eta)\}^\kappa \in \mathrm{BV}(\lambda_1,\infty).$$

By Lemma 2, $\mathscr{F}_2$ can be expressed as a sum of constant multiples of integrals of the type

$$\int_{\lambda_1}^{\eta} C_\lambda^\kappa(\sigma)\{\phi(\eta)-\phi(\sigma)\}^{\kappa-\kappa'}\{\phi^{(1)}(\sigma)\}^{\beta_1}\{\phi^{(2)}(\sigma)\}^{\beta_2}\ldots\{\phi^{(\kappa+1)}(\sigma)\}^{\beta_{\kappa+1}}\,d\sigma,$$

where the β's are positive integers or zeros such that

$$\beta_1+\beta_2+\ldots+\beta_{\kappa+1} = \kappa' \leqslant \kappa$$

and
$$\beta_1+2\beta_2+\ldots+(\kappa+1)\beta_{\kappa+1} = \kappa+1.$$

Expanding $\{\phi(\eta)-\phi(\sigma)\}^{\kappa-\kappa'}$ by the binomial theorem, we easily see that $\mathscr{F}_2/\{\phi(\eta)\}^\kappa$ is a sum of constant multiples of terms of the type

$$\frac{1}{\{\phi(\eta)\}^r}\int_{\lambda_1}^{\eta}\left(\frac{C_\lambda^\kappa(\sigma)}{\sigma^\kappa}\right)\sigma^\kappa\{\phi(\sigma)\}^\alpha\{\phi^{(1)}(\sigma)\}^{\alpha_1}\ldots\{\phi^{(\kappa+1)}(\sigma)\}^{\alpha_{\kappa+1}}\,d\sigma,$$

where the α's are positive integers or zeros such that

$$0 < \alpha+\alpha_1+\ldots+\alpha_{\kappa+1} = r \leqslant \kappa$$

and
$$\alpha_1+2\alpha_2+\ldots+(\kappa+1)\alpha_{\kappa+1} = \kappa+1.$$

Since, by hypothesis,

$$C_\lambda^\kappa(\sigma)/\sigma^\kappa \in \mathrm{BV}(\lambda_1, \infty),$$

the result follows at once by the application of Lemmas 4 and 3.

2.4. In this section I show that, if $\phi(t)$ is a logarithmico-exponential function ('L-function') such that, for some Δ,

$$\phi(t) = O(t^\Delta) \quad (t \text{ large}), \tag{2.41}$$

then the conditions of the theorem are necessarily satisfied.

Since any L-function is ultimately monotonic, it is sufficient for our purpose to prove that

$$\frac{t^r \phi^{(r)}(t)}{\phi(t)} = O(1) \quad (t \text{ large};\ r = 1, 2, \ldots, \kappa). \tag{2.42}$$

Hirst (4) has shown that any L-function satisfying (2.41) satisfies the conditions of his theorem, and that any function satisfying the conditions of his theorem satisfies (2.42).

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ON ORBITAL TOPOLOGIES

By R. W. BAGLEY (*University of Kentucky*)

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In a paper by D. O. Ellis (**1**) the notion of orbital topologies is defined as follows: let S be an infinite set and γ a mapping of S into itself. A point x in S is an accumulation point of $A \subset S$ if $\gamma^{n_y}(y) = x$ for infinitely many y in A with n_y a non-negative integer. With closure defined in the usual manner, Ellis shows that this yields a T_1 space. This space is denoted by S_γ. The purpose of this paper is to solve some problems mentioned in (**1**) and consider the special case in which γ is biuniform and onto.

Notation. A', A^0, $C(A)$ will denote respectively accumulation points of A, the interior of A, and the complement of A.

THEOREM 1. *S_γ is a T_2 space if and only if each accumulation point is a fixed point.*

Proof. In (**1**) it is proved that the condition is necessary for S_γ to be T_2. To prove that it is sufficient, we take two points of accumulation x_0, x_1 and for each $x \in S_\gamma$ define a set as follows:

$$A_x = \{y \in S \mid \gamma^n(y) = x \text{ for some non-negative integer } n\}.$$

Since x_0 and x_1 are fixed points, we see that A_{x_0} and A_{x_1} are disjoint. For each $x \in S_\gamma$ the set A_x is open. This proves that S_γ is T_2.

THEOREM 2. *A necessary and sufficient condition that γ be continuous is that whenever $\{\gamma^{-1}(x_0)\}$ is infinite x_0 is fixed.*

Proof. Suppose γ continuous and $A = \{\gamma^{-1}(x_0)\}$ infinite. Then A is closed and, since $x_0 \in A'$, we have $x_0 \in A$ or $\gamma(x_0) = x_0$. In order to prove that the condition is sufficient, we let B be any closed set and take $x \in \{\gamma^{-1}(B)\}'$. Then for infinitely many y in $\gamma^{-1}(B)$ we have $\gamma^{n_y}(y) = x$. If the set $C = \{\gamma(y) \mid y \in \gamma^{-1}(B), \gamma^{n_y}(y) = x\}$ is infinite, then $\gamma(x) \in B'$, where B' is contained in B. If C is finite, then, for some $y_0 \in \gamma^{-1}(B)$ with $\gamma^{n_{y_0}}(y_0) = x$, we have $\{\gamma^{-1}(y_0)\}$ infinite. Therefore $\gamma(y_0) = y_0$. Now

$$x = \gamma^{n_{y_0}}(y_0) = y_0$$

and $x \in \gamma^{-1}(B)$. Thus $\gamma^{-1}(B)$ is closed and γ is continuous.

We note that, if γ is biuniform and onto, then S_γ is T_2 only when discrete. We see later that, for γ biuniform and onto, S_γ is discrete if and only if γ is periodic at each point of S (Theorem 3, Cor. 1).

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LEMMA 1. *For γ biuniform and onto, A_{x_0} (as defined in Theorem 1) is finite if and only if γ is periodic at x_0.*

Proof. Suppose that γ has period n at x_0 and let

$$\gamma(x_i) = x_{i-1} \quad (i = 1,\ldots,n).$$

Then $\gamma^i(x_i) = x_0$. Let x be any element such that $\gamma^r(x) = x_0$, where $r > n$. Now $r = kn+r_0$ with $0 \leqslant r_0 < n$. We have

$$\gamma^{kn}(x_0) = x_0 = \gamma^r(x).$$

Thus $x_0 = \gamma^{r_0}(x)$ and $x = x_{r_0}$, which completes the proof that A_{x_0} is finite.

Now suppose that A_{x_0} is finite. Then

$$A_{x_0} = \{x_i\}_{i=0}^{n},$$

where $\gamma^i(x_i) = x_0$. Let $x = \gamma^{-1}(x_n)$. Then

$$\gamma^{n+1}(x) = \gamma^n(x_n) = x_0.$$

Thus $x = x_{i_0}$, where $0 \leqslant i_0 \leqslant n$ and $\gamma^{i_0}(x) = x_0$. Then

$$\gamma^{n+1-i_0}(x_0) = \gamma^{n+1}(x) = x_0,$$

which proves that γ is periodic at x_0.

THEOREM 3. *If γ is biuniform and onto, then A_{x_0} is dense in itself if and only if γ is not periodic at x_0.*

Proof. Suppose first that γ is not periodic at x_0. Let $y_0 \in A_{x_0}$ $(y_0 \neq x_0)$. Then $\gamma^n(y_0) = x_0$ for some $n \geqslant 1$, and, by the lemma, for infinitely many y we have $\gamma^{n_y}(y) = x_0$, where $n_y > n$. Thus

$$\gamma^n(y_0) = \gamma^{n_y}(y), \qquad y_0 = \gamma^{n_y-n}(y),$$

which shows that $y_0 \in A'_{x_0}$. Therefore, A_{x_0} is dense in itself since x_0 is obviously an accumulation point of A_{x_0}. Now, if A_{x_0} is dense in itself, since it is non-null it must be infinite, which means that γ is not periodic at x_0.

COROLLARY 1. *If γ is biuniform and onto, then x_0 is an accumulation point of S_γ if and only if γ is not periodic at x_0.*

COROLLARY 2. *If γ is biuniform and onto, then S_γ is dense in itself if and only if γ is not periodic at any point of S.*

In (**1**) it was shown that a necessary condition for S_γ to be compact is that each orbit $\Gamma(x) = \{\gamma^n(x)\}$ $(n = 0, 1,\ldots)$ be finite. To see that this condition is not sufficient, let γ be biuniform and onto and such that S_γ is discrete. Moreover, for any γ biuniform, S_γ is not compact. To see this, let $x_0 \in S'_\gamma$ and note then that $\Gamma(x_0)$ is infinite since γ is not periodic at x_0.

THEOREM 4. *If γ is biuniform and onto, then S_γ is the point-set sum of two sets A and B (A is null if and only if S_γ is dense in itself and B null if and only if S_γ is discrete), where A is discrete and B is the point-set sum of perfect countably infinite sets pairwise disjoint.*

Proof. Let $\{\alpha\}$ be a well-ordered set having the same cardinal as S'_γ. For $x_\alpha \in S'_\gamma$, define

$$B_{x_\alpha} = \bigcup_{n=1}^{\infty} \Gamma(x_n^\alpha),$$

where $\gamma^n(x_n^\alpha) = x_\alpha$. For $x_{\alpha'} \in S'_\gamma - \bigcup_{\alpha<\alpha'} B_{x_\alpha}$, we define the set $B_{x_{\alpha'}}$ as before. We let $B = \bigcup_\alpha B_{x_\alpha}$ and $A = S_\gamma - B$. We note by the structure of B_{x_α} that γ is a homeomorphism of S_γ onto itself.

Let J be the set of integers with the topology obtained by taking, as closed sets, the set of all the integers and those subsets which are bounded on the left. The following theorem characterizes S_γ among general topological spaces when γ is biuniform and onto.

THEOREM 5. *Let S be a topological space. The topology on S is an orbital topology induced by a biuniform onto mapping γ if and only if $S = A \cup B$, where A is discrete and $B = \bigcup_\alpha B_\alpha$ (the sets B_α are assumed to be pairwise disjoint) with each B_α homeomorphic to J.*

Proof. For $x \in A$, define $\gamma(x) = x$. If $x \in B_\alpha$, then x corresponds to some integer n under the homeomorphism of the theorem. We define $\gamma(x) = y$, where y corresponds to $n+1$ under this homeomorphism. The remainder of the proof follows from the structure of the sets B_{x_α}. We note that γ is not unique.

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ON CONTINUOUS SEMI-INDEPENDENT FUNCTIONS

By H. M. SENGUPTA (*Calcutta*)

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In an interesting paper entitled 'Note on continuous independent functions' Offord (**1**) determined a sufficient condition that two functions continuous on $0 \leqslant x \leqslant 1$ and not constant in it may fail to be independent.

The independence of two real functions $f(x)$ and $g(x)$ on $0 \leqslant x \leqslant 1$ is defined as follows. Consider all sets

$$E_1 = E_x[\alpha \leqslant f(x) \leqslant \beta] \quad \text{and} \quad E_2 = E_x[\alpha' \leqslant g(x) \leqslant \beta'],$$

where α, β and α', β' are any pairs of real numbers between the lower and upper bounds l, L and m, M of the functions $f(x)$ and $g(x)$ respectively. The functions $f(x)$ and $g(x)$ are said to be *independent* if

$$|E_1 E_2| = |E_1| . |E_2|.$$

Offord established the following theorem:

Offord's Theorem. *Supposing $f(x)$ and $g(x)$ both to be continuous on the closed interval I $(0 \leqslant x \leqslant 1)$ and neither to be a constant, then, if either function assumes at least one of its values at most enumerably often, $f(x)$ and $g(x)$ cannot be independent.*

In (**2**) I effected a small extension of Offord's theorem; this extension is explained below.

Let a denote one of the values of $g(x)$ and let $E_{g=a} \equiv E_x[g(x) = a]$ be the set of points in $0 \leqslant x \leqslant 1$ for which $g(x) = a$. We define by $F_a \equiv f[E_{g=a}]$ the set of values assumed by $f(x)$ at the points $E_{g=a}$, i.e. at the points in $0 \leqslant x \leqslant 1$, where $g(x) = a$. The extension of Offord's theorem may now be enunciated in the form of the following theorem.

Theorem. *If the set F_a be not identical with the entire range of values of $f(x)$ for any value a of $g(x)$, then the functions $f(x)$ and $g(x)$ cannot be independent.*

In the same paper I also proved the following.

If $f(x)$ and $g(x)$ be continuous in $0 \leqslant x \leqslant 1$ and be not constant in it, then a necessary and sufficient condition for $|E_1 E_2| > 0$ is that the set of values assumed by $f(x)$ over the set of points where $g(x)$ has the

value a $(m \leqslant a \leqslant M)$, must be identical with the entire interval $l \leqslant x \leqslant L$ and also that the set of values assumed by $g(x)$ over the set of points where $f(x) = a'$ $(l \leqslant a' \leqslant L)$ must be identical with the entire interval $m \leqslant y \leqslant M$.

Pairs of functions $f(x)$ and $g(x)$, continuous on $0 \leqslant x \leqslant 1$ and not constant in it, will be referred to as *continuous and semi-independent* in $0 \leqslant x \leqslant 1$ if $|E_1 E_2| > 0$.

The purpose of the present note is to prove theorems 1 and 2.

THEOREM 1. *Two functions $f(x)$ and $g(x)$, continuous and semi-independent in $0 \leqslant x \leqslant 1$ and not constant in it, furnish a continuous representation on a rectangular area of the unit linear interval.*

Proof. Let $f(t)$ and $g(t)$ be continuous and semi-independent in the interval $0 \leqslant x \leqslant 1$. Let none of the functions be constant in the interval. Further, let l, L and m, M be the lower and upper bounds of $f(t)$ and $g(t)$ respectively on $0 \leqslant t \leqslant 1$. Then $l < L$ and $m < M$. Draw the rectangle $l \leqslant x \leqslant L$, $m \leqslant y \leqslant M$ in the (x, y)-plane and take a unit length $0 \leqslant t \leqslant 1$.

Let (c, d) be any point in the above closed rectangle which I shall call R in the sequel. Since $l \leqslant c \leqslant L$, the value c is assumed by $f(t)$. Denote the set of points on $0 \leqslant t \leqslant 1$ for which $f(t) = c$ by the symbol

$$T_{f=c} \equiv E_t[f(t) = c].$$

The set $T_{f=c}$ is closed, non-dense, and non-null. Now over the set $T_{f=c}$ the function $g(t)$ assumes every value in $m \leqslant y \leqslant M$. And, since $m \leqslant d \leqslant M$, there is a closed non-null set of points $T_{g=d}$ on the unit interval $0 \leqslant t \leqslant 1$, where $g(t) = d$ and the set of points $T_{g=d}$ has at least one point in common with $T_{f=c}$. Hence the intersection $T_{f=c} \cdot T_{g=d}$, which is a closed set, is non-null; and over this set $f(t) = c$ and $g(t) = d$. It follows that to any point (c, d) of the rectangle R corresponds a non-null closed set of points $T_c \cdot T_d$ on the unit interval; and to two distinct points of R correspond two distinct non-null closed sets of points on the linear interval $0 \leqslant t \leqslant 1$.

On the other hand, for any t in $0 \leqslant t \leqslant 1$, $f(t)$ lies in the interval $l \leqslant x \leqslant L$ and $g(t)$ lies in $m \leqslant y \leqslant M$. So to any point t of the linear interval $0 \leqslant t \leqslant 1$ corresponds a unique point of R. Thus any pair of non-constant, continuous semi-independent functions on $0 \leqslant t \leqslant 1$ give a continuous representation of a rectangular area on a linear interval.

In particular, as the aggregate of pairs of continuous independent functions is a subset of the aggregate of pairs of continuous semi-independent functions, it follows that a pair of continuous independent

functions on a linear interval furnishes a continuous representation of a rectangular area on the linear interval.

Since a continuous representation of a rectangular area on a linear interval cannot be effected in a (1, 1) manner, it follows that there is at least one pair of numbers c and d ($l \leqslant c \leqslant L$, $m \leqslant d \leqslant M$) such that $T_{f=c}.T_{g=d}$ consists of more than one point.

The converse of the above theorem is also true. It is interesting to note that we can prove the following theorem.

THEOREM 2. *Any continuous representation of a rectangle on a linear interval defines a pair of continuous semi-independent functions on the interval.*

Proof. Let a correspondence of the points of the rectangle $l \leqslant x \leqslant L$, $m \leqslant y \leqslant M$ be effected on the linear interval $0 \leqslant x \leqslant 1$. Suppose

$$x = x(t), \qquad y = y(t)$$

to be the functions which effect the correspondence and suppose that both $x(t)$ and $y(t)$ are continuous on $0 \leqslant t \leqslant 1$.

Take any value c of $x(t)$ and, as before, let $T_{f=c}$ be the set of points of $0 \leqslant t \leqslant 1$ at which $x(t) = c$. This set is closed and, since the points (c, y) where $m \leqslant y \leqslant M$ belong to the rectangle, the set of values of $y(t)$ on the set $T_{f=c}$ runs over the entire range of values of $y(t)$. And this is true for every value c of $x(t)$ in the range $l \leqslant x \leqslant L$. The same argument applies the other way about, i.e. if d be a value of $y(t)$ in the interval $m \leqslant y \leqslant M$, then the set of values assumed by $x(t)$ over the set where $y(t) = d$ coincides with the entire range of values of $x(t)$. Accordingly the pair of functions $x(t)$ and $y(t)$ are continuous and semi-independent in $0 \leqslant t \leqslant 1$.

In particular, Peano's curve furnishes a pair of continuous and semi-independent functions on the unit interval.

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THE FUNDAMENTAL GROUP OF TWO SPACES WITH A COMMON POINT

By H. B. GRIFFITHS (*Bristol*)

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1. It is well known that, if two finite connected complexes K, L have exactly one point in common, then the fundamental group of their union is the free product $\pi_1(K) \circ \pi_1(L)$. Also, the first singular homology group over the integers is the direct sum $\mathscr{H}^1(K)+\mathscr{H}^1(L)$. For use elsewhere, it was found necessary to extend these results to a similar union of more general spaces, and these extensions are completed in Theorem 1 and Corollary 2.5 below. For the theorem, we have to assume that at least one of the spaces is† $1-LC$ at the common point, and I show by an example in § 3 that the $1-LC$ condition cannot in general be relaxed. Our example is a space of the form $Z = X \cup Y$, where $X \cap Y$ is a single point, with $\pi_1(X) = \pi_1(Y) = 0$, yet such that the cardinal of each of $\pi_1(Z)$ and $\mathscr{H}^1(Z)$ is at least that of the continuum. The result for $\mathscr{H}^1(Z)$ is interesting because the one-dimensional Čech homology group of Z over the integers is zero. The general homotopy result is in contrast to the corresponding result for the higher homotopy groups, where no local condition at the common point is required [see e.g. Hilton (**4**) 42; my results correct the statement there at the top of p. 43]. Moreover, my result for $\mathscr{H}^1(Z)$ shows that the theorem for higher dimensions, given by Hilton (**4**), cannot follow purely from the Abelian property of the homotopy groups of dimension exceeding 1. Consideration of the example Z uses the 'topologist's product' of a sequence of groups, which is closely related to Higman's unrestricted free product (**3**); and in this connexion I prove in § 4 the purely algebraic Theorems 5 and 6.

Notation. I usually denote the unit segment in R^1 by E^1, but sometimes to avoid ambiguity this segment will be denoted by I. If σ is any interval (open or not) in R^1, we use $\dot{\sigma}$ to denote its frontier. Let G_1, G_2, G be three groups, with homomorphisms h_r: $G_r \to G$ $(r = 1, 2)$. Then we define a homomorphism $(h_1 \wedge h_2)$ of the free product $G_1 \circ G_2 \to G$ by

$$(h_1 \wedge h_2)(a_1 b_1 \ldots a_m b_m) = (h_1 a_1)(h_2 b_1)\ldots(h_1 a_m)(h_2 b_m),$$

where $a_i \in G_1$, $b_i \in G_2$ $(i = 1, \ldots, m)$. Since every element of $G_1 \circ G_2$ can be expressed uniquely‡ as a product of non-zero elements from G_1 and

† See, for example, Wilder (**8**), IX 7.9, 287, for the definition.

‡ See, for example, Van der Waerden (**6**).

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G_2, the above definition of $h_1 \wedge h_2$ is unambiguous. We also define a homomorphism h_1+h_2 of the direct product G_1+G_2 in G by

$$(h_1+h_2)(g_1+g_2) = (h_1 g_1)+(h_2 g_2).$$

We shall be concerned throughout with two Hausdorff spaces X_1, X_2 which satisfy† the first axiom of countability [Hausdorff (**2**) 239 (9)] and which have precisely one point x in common. Let $X_1 \cup X_2 = X$; we shall suppose that each X_i is *closed*‡ in X. Define

$$i_r\colon \pi_1(X_r, x) \to \pi_1(X, x) \quad (r = 1, 2)$$

to be the injection homomorphism. From now on, I shall often omit the base point and write e.g. $\pi_1(X)$ for $\pi_1(X, x)$. Then the main theorem can be stated thus:

THEOREM 1. *If one of* X_1, X_2 *is* $1-LC$ *at* x, *then*

$$(i_1 \wedge i_2)\colon \pi_1(X_1) \circ \pi_1(X_2) \approx \pi_1(X).$$

2. The theorem is an immediate consequence of the following two theorems:

THEOREM 2. *If* X_1 *is* $1-LC$ *at* x, *then* $(i_1 \wedge i_2)$ *is 'onto'*.

2.1. *Proof.* Let $f\colon E^1, \dot{E}^1 \to X$, x be a given mapping. Since E^1 is compact and f is continuous, then $F = f(E^1)$ is closed in X; hence, since X_r is closed in X $(r = 1, 2)$, then $F_r = F \cap X_r$ is closed in X_r. Therefore $F_r^1 = f^{-1}(F_r)$ is closed in E^1, so that the open set $E^1 - F_2^1$ is the union of a countable set S of disjoint intervals I, each I being open in E^1.

Because the space X_1 satisfies the first countability axiom and is $1-LC$ at x, there exists in X_1 a sequence

$$X_1 \supseteq U_1 \supseteq \ldots \supseteq U_m \supseteq \ldots$$

of neighbourhoods of $x \in X_1$, such that

(i) $$\bigcap_{m=1}^{\infty} U_m = x,$$

and, for each $m > 1$, the injection

(ii) $$j_m\colon \pi_1(U_m) \to \pi_1(U_{m-1})$$

is trivial. For each m, let S_m be the collection of all those $I \in S$ for which $f(I) \subseteq U_m$. Since f is uniformly continuous on E^1, then

$$S_m - S_{m+1} = \{I_{m,1}, I_{m,2}, \ldots, I_{m,p(m)}\}$$

is a finite set. Also each $\dot{I}_{mn} \subseteq F_2 \cap F_1$, so that $f(\dot{I}_{mn}) = x$; thus

$$f_{mn} = (f \mid I_{mn})$$

† I have no examples to show the necessity of this restriction; I merely require it to make my proofs work.

‡ Otherwise Theorem 1 is false, as D. Puppe has pointed out to me by means of the example of a circle divided into a closed arc and a half-open one.

defines an element of $\pi_1(U_m, x)$ and therefore by (ii), if $m > 1$, then f_{mn} is $\simeq 0$ rel x† in U_{m-1}, say by a homotopy

$$\psi_{mn}\colon (\bar{I}^1_{mn} \times l, \dot{I}_{mn} \times l) \to (U_{m-1}, x).$$

Every $\xi \in E^1$ is either in F_2 or in some (unique) I_{mn}, and so we can define without ambiguity a homotopy

$$\psi\colon (E^1 \times l, \dot{E} \times l) \to (X, x) \tag{2.11}$$

by $$\psi(\xi, t) = \begin{cases} f(\xi) & \text{if } \xi \in (F_2 \cup I_{1,1} \cup \ldots \cup I_{1,p(1)}), \\ \psi_{mn}(\xi, t) & \text{if } \xi \in I_{mn} \quad (m > 1). \end{cases}$$

We must prove ψ continuous at all points $z \in E^1 \times l$. The continuity is obvious at all z except when $z \in \dot{E}^1 \times l$, and here it suffices by (i) to prove that, given U_q, there exists a neighbourhood $V = V(z) \subseteq E^1 \times l$ such that $\psi(V) \subseteq U_q$. If the point $\xi_0 = 0$ is in some $I_{m,n}$, then the continuity of ψ for all $z = (0, t)$ follows from that of ψ_{mn}. Suppose then that ξ_0 is in no interval $\bar{I}$, where $I \in S$. Now f is continuous at ξ_0, so that there is a neighbourhood $W = W(\xi_0) \subseteq E^1$ for which $f(W) \subseteq U_q$. The uniform continuity of f on E^1 also implies that

$$\lim_{m \to \infty} \left\{ \text{l.u.b.}_{1 \leqslant n \leqslant p(m)} (\text{length } I_{mn}) \right\} = 0;$$

hence we can assume W to be such that every I_{mn} which meets W is contained in W and is so small that $\psi_{mn}(I_{mn} \times l) \subseteq U_q$. Hence, if $\xi \in W$, then either $\xi \in F_2$ or ξ is in some I_{mn}. If $\xi \in F_2$, then $\psi(\xi, t) = f(\xi)$ for all t, and

$$f(\xi) \in f(W) \subseteq U_q;$$

and, if $\xi \in I_{mn}$, then $\xi \in W$, whence

$$\psi(\xi, t) \in \psi_{mn}(I_{mn} \times l) \subseteq U_q.$$

Therefore $\psi(W \times l) \subseteq U_q$, i.e. ψ is continuous at all points $(0, t) \in E^1 \times l$. Similarly ψ is continuous at all points $(1, t)$, so that ψ is continuous everywhere in $E^1 \times l$ as required.

Clearly $\psi(\xi, 0) = f(\xi)$; define f' by $f'(\xi) = \psi(\xi, 1)$, so that $f' \simeq f$ rel x in X. We now express E^1 in the obvious way as

$$E^1 = J_1 \cup I_{11} \cup J_2 \cup I_{12} \cup \ldots \cup J_p \cup I_{1p} \cup J_{p+1},$$

where $p = p(1)$, the J's are closed intervals disjoint from the I's and each other, and the numbering is such that, if

$$\zeta \in J_i, \qquad \eta \in I_{1i}, \qquad \xi' \in J_{i+1},$$

then $\xi < \eta < \xi'$ for each appropriate i. Write

$$f_j = f' | \bar{I}_{1j}, \qquad g_j = f' | J_j.$$

† i.e. with x kept fixed during the homotopy.

Then $f_j(\bar{I}_{1j}, \dot{I}_{1j}) \leqslant (X_1, x)$ and $g_j(J_j, \dot{J}_j) \leqslant (X_2, x)$, so that f' is the product mapping

$$f' = g_1 f_1 g_2 f_2 \ldots g_p f_p g_{p+1}.$$

Therefore, if homotopy classes in $\pi_1(X, x)$, $\pi_1(X_r, x)$ be denoted by $R(.)$, $R_r(.)$, we have

$$\begin{aligned} Rf = Rf' &= R(g_1 f_1 g_2 f_2 \ldots g_p f_p g_{p+1}) \\ &= (Rg_1)(Rf_1)\ldots(Rg_p)(Rf_p)(Rg_{p+1}) \\ &= (i_2 R_2 g_1)(i_1 R_1 f_1)\ldots(i_2 R_2 g_p)(i_1 R_1 f_p)(i_2 R_2 g_{p+1}) \\ &= (i_1 \wedge i_2)\{(R_2 g_1)(R_1 f_1)\ldots(R_2 g_p)(R_1 f_p)(R_2 g_{p+1})\} \\ &= (i_1 \wedge i_2)\delta, \text{ say,} \end{aligned}$$

where $\delta \in \pi_1(X_1) \circ \pi_1(X_2)$. Therefore $(i_1 \wedge i_2)$ is 'onto', as required.

It should be noted that the next theorem does not require the $1-LC$ property.

THEOREM 3. *The homomorphism $i_1 \wedge i_2$ always has zero kernel.*

2.2. *Proof.* Every element γ in $\pi_1(X_1) \circ \pi_1(X_2)$ can be written uniquely† in the form

$$\gamma = \alpha_1 \beta_1 \alpha_2 \beta_2 \ldots \alpha_m \beta_m, \tag{i}$$

where $\alpha_j \in \pi_1(X_1)$, $\beta_j \in \pi_1(X_2)$ $(j = 1, \ldots, m)$, and where, without loss of generality, we can assume that no α or β is zero except possibly β_m. We are required to prove that, if $\gamma \neq 0$, then

$$(i_1 \wedge i_2)\gamma \neq 0. \tag{ii}$$

Let $\theta_r\colon X, x \to X_r, x$ $(r = 1, 2)$ be mappings defined by $\theta_r(\xi) = \xi$ if $\xi \in X_r$ and $\theta_r(\xi) = x$ otherwise. Then, since X_r is closed in X, θ_r is clearly continuous and induces a homomorphism

$$\theta_r'\colon \pi_1(X) \to \pi_1(X_r) \tag{iii}$$

such that $\theta_r' i_r$ is the identity.

First we assert that i_r is univalent; for, if $\delta \in \pi_1(X_r)$ and $i_r(\delta) = 0$, then $0 = \theta_r' i_r(\delta) = \delta$. Next, we may assume that $\beta_2 \neq 0$; otherwise $\gamma = \alpha_1 \beta_1 \alpha_2$ and, if $(i_1 \wedge i_2)\gamma = 0$, then

$$0 = \theta_2(i_1 \wedge i_2)\gamma = i_2(\beta_1),$$

so that, since i_2 is univalent, $\beta_1 = 0$ contrary to the fact. We shall therefore prove the theorem by assuming that α_1, α_2, β_1, β_2 at least are non-zero, and yet that (ii) is false; this will lead to a contradiction.

Let Π be a regular polygonal disk in the plane R^2, with edges

$$a_1, b_1, \ldots, a_m, b_m \quad \text{(in that order).}$$

† See, for example, Van der Waerden (6).

We orient the edges and then represent

$$i_1\alpha_j, \quad i_2\beta_j \quad (j = 1,..., m)$$

by maps

(iv) $$f_j: a_j, \dot{a}_j \to X, x, \qquad g_j: b_j, \dot{b}_j \to X, x,$$

respectively, where, if $\beta_m = 0$, $g_m(b_j) = x$. These taken together form a mapping ψ of the boundary of Π into X. By our assumption that (ii) is false, ψ has an extension ψ: $\Pi \to X$ over the whole interior Π_0 of Π.

Let $F = \psi^{-1}(x)$. Since ψ is continuous, the set $\Pi - F$ is open in Π; and therefore, since an open plane set is locally connected, each component C of $\Pi - F$ is open in Π. Hence C is an open, connected, and locally connected set with compact separable closure in Π. Therefore, by Wilder [(8), 3.11, 81], C is arc-wise connected. For each $j = 1,..., m$ let

$$A_j = a_j \cup \bigcup C - \dot{a}_j,$$

where C runs through all components C of $\Pi - F$ for which $C \cap a_j \neq 0$. Define B_j relative to b_j similarly except that, if $\beta_m = 0$, then B_m is defined to be the empty set. Then A_j, B_j are obviously arc-wise connected and all are non-empty except possibly B_m. I shall shortly prove:

[2.21] *For no p, q can we have $A_p = B_q$.*

[2.22] *Either one non-empty B_j is distinct from the rest of the B_i, or one A_h is distinct from the rest of the A_i.*

Granted these for the moment, the reader may find helpful the following outline of the proof, given for the case $m = 4$ with the aid of Figs. 1, 2. The proof for the general case should then be clearer. Suppose then that, as in Fig. 1, A_4 is distinct from the other A_i. The proof will show that there is a Jordan curve containing the edge a_4 such that, given $\epsilon > 0$, $A_4 - a_4$ is contained in the interior of J but $J \subseteq U(A_4, \epsilon)$. Let μ be the portion of J indicated by $v_1 u_1 u_2 v_2$ in Fig. 2, so that $J = a_4 + \mu$. Then, by the 'distinctness' of A_4, a_4 is deformable in Π to μ with v_1, v_2 fixed; and therefore with f_4 as in (iii), we have

(v) $$\psi \mid \mu = \psi \mid a_4 = f_4 \operatorname{rel} x \text{ in } X.$$

We then obtain a sequence of curves μ_n like μ, each with end-points v_1, v_2 and with $\mu_n \subseteq U(P, \epsilon_n)$, where $P = \mathfrak{F}A_4 - a_4$ and $\epsilon_n \to 0$. Hence $\{\psi(\mu_n)\}$ is a sequence of loops in X, all converging to x. But, in $U(P, \epsilon_n)$, μ_n is deformable, with v_1, v_2 fixed, to μ_{n+1}; whence it will follow that, rel x,

$$\psi \mid \mu_1 \simeq \psi \mid \mu_2 \simeq \ldots \simeq \psi \mid \mu_n \simeq \ldots \simeq x \quad \text{in } X.$$

Hence, with (v), we obtain $f_4 \simeq 0 \operatorname{rel} x$ in X; thus $i_1\alpha_4 = 0$, whence

$\alpha_4 = 0$ since i_1 is univalent. This contradicts the assumption about α_4 in (i).

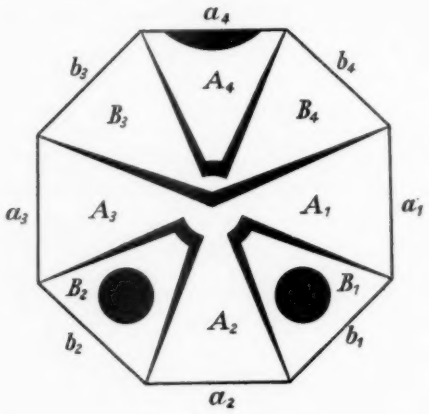

FIG. 1.

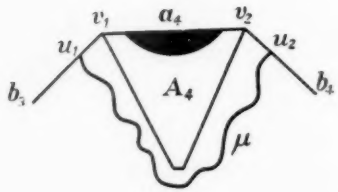

FIG. 2.

We now proceed to the general proof.

Proof of [2.21]. If, contrary to [2.21], we had $A_p = B_q$, then there would be, in the non-empty, arc-wise connected set A_p, an arc λ with end-points $\xi \in a_p$, $\eta \in b_q$ such that

$$\lambda - \xi - \eta \subseteq \Pi_0 \cap A_p, \qquad \psi(\xi) \in X_1 - x, \qquad \psi(\eta) \in X_2 - x.$$

But the removal of x disconnects X, so that the connected set $\psi(\lambda)$ in X meets x in order to connect its own end-points. Hence, $\lambda \cap \Pi_0$ meets F, contrary to the fact that $A_p \cap \Pi_0 \subseteq \Pi_0 - F$. This contradiction establishes [2.21]. Note that, if $A_p \cap B_q \neq 0$, then $A_p \cup B_q$ is arc-wise connected. Hence we could construct the arc λ as before and obtain a contradiction. Thus, for all p, q,

(vi) $$A_p \cap B_q = 0.$$

Proof of [2.22]. If there is no non-empty B_j distinct from the rest of the B_i, then, since $m \geqslant 2$, there is a least $d(j) \neq j$ such that $B_j = B_{d(j)} \neq 0$. We assert that† either $d(1) = m$, or else for some j, $|d(j)-j| = 1$. If $d(1) \neq m$, then $m > d(1) > 1$; suppose that $d(1)-1 > 1$. If $d(2) = 1$, the assertion holds; if $d(2) \neq 1$, then, since $d(2) \neq 2$, we have

$$d(2)-2 \geqslant 1.$$

But, since $b_i' = b_i - \dot{b}_i \subseteq B_i$ for all non-empty B_i, then $b_1', b_{d(1)}' \subseteq B_1$. Hence the arc-wise connected set B_1 spans the boundary of Π. Since Π is a closed 2-cell, then B_1 separates Π, and, since $d(2) \neq 1$, then, by the 'least' property of $d(2)$, we have $B_{d(2)} = B_2 \neq B_1$. Therefore B_2 lies in the component of $\Pi - B_1$ containing b_2. Hence $d(2) < d(1)$, so that $d(2)-2 < d(1)-1$. By repeating this argument at most $d(1)-1$ times, we obtain a decreasing sequence of integers

$$d(1)-1 > d(2)-2 > \ldots > d(h)-h = 1.$$

Hence $B_h = B_p$, say, where $p = d(h) = h+1$ if $h \neq 1$, and $p = 2$ or m otherwise. If $p = m$, then B_m is not empty since $B_1 \neq 0$.

We now obtain a contradiction by supposing not only that each non-empty B_j is equal to some $B_{d(j)}$ with $d(j) \neq j$, but also that every A_j is equal to some $A_{k(j)}$ with $k(j) \neq j$ (as is possible since $m \geqslant 2$).

With p as above, then in particular $A_p = A_q$, where $q = k(p) \neq p$; and hence $a_p' \cup a_q' \subseteq A_p$, where $a_j' = a_j - \dot{a}_j$ $(j = 1,\ldots,m)$. With h as above, then $b_h' \cup b_p' \subseteq B_h$. Hence there lie arcs λ, μ in A_p, B_h, respectively, connecting the edges a_h, a_q and the edges b_h, b_p. But then μ spans the closed 2-cell Π, and so separates Π. Hence λ intersects μ since a_p is the only a-edge of Π which lies in the same component of $\Pi - \mu$ as points of b_h, b_p, because b_h, b_p are consecutive b-edges separated by a_p. However, if λ intersects μ, then $A_p \cap B_h \neq 0$ contrary to (vi). This contradiction establishes [2.22].

Let us therefore assume that, say, A_t is distinct from the rest of the A_i. The closure A of A_t in R^2 is a connected compact set to which we can apply Zoretti's theorem [see e.g. Whyburn (7) VI 3.11]. Thus, given $\epsilon > 0$, there is a simple closed curve $J \subseteq U(A, \epsilon) - A$ which encloses A. Let v_1, v_2 be the end points of a_t, and choose points u_1, u_2 common to J and the boundary of Π such that‡ $\rho(u_r, v_r) < \epsilon$ $(r = 1, 2)$. Let $u_r v_r$ denote the segment joining u_r to v_r, so that $u_r v_r \subseteq U(A, \epsilon) \cap \Pi$. Hence, if λ is the arc of $J \cap \Pi$ with end-points u_1, u_2, we have

(vii) $$\mu_\epsilon = v_1 u_1 + \lambda + u_2 v_2 \subseteq U(A, \epsilon) \cap \Pi.$$

† As suggested by Fig. 1.

‡ ρ denotes the metric in R^2.

Let $p_\epsilon = \psi \mid \mu_\epsilon$ so that $p_\epsilon(\dot{\mu}_\epsilon) = \psi(\dot{a}_t) = x$. Therefore p_ϵ defines an element of $\pi_1(X, x)$. By the definition of f_t in (iv) we have $f_t \not\simeq 0 \operatorname{rel} x$ in X. Also μ_ϵ is deformable to a_t, with fixed end-points, in Π; and, since $A_t \neq A_s$ for all $s \neq t$, then a_t is the only edge of Π contained in A. Hence

$$p_\epsilon \simeq f_t \operatorname{rel} x \text{ in } X,$$

and therefore, with θ_1 as in (iii),

(viii) $$f_t = \theta_1 f_t \simeq \theta_1 p_\epsilon \operatorname{rel} x \text{ in } X_1.$$

Given a neighbourhood V of x in X_1, then, since $\theta_1\psi\colon \Pi \to X$ is continuous, there exists in Π a δ-neighbourhood W of F such that $\theta_1\psi(W) \subseteq V$. If, therefore, in (vii) we take $\epsilon = \frac{1}{2}\delta$, then, in (viii),

$$\theta_1 f_\epsilon(\mu_\epsilon, \dot{\mu}_\epsilon) \subseteq (V, x),$$

because $$\mu_\epsilon \subseteq U(A, \epsilon) - A \subseteq U(\mathscr{F}A - a_t, \epsilon) \subseteq U(F, \epsilon)$$

since $\mathscr{F}A_t - a_t \subseteq F$ (U now refers to neighbourhoods in Π).

Since X_1 satisfies the first countability axiom at x, there is a sequence $\{V_n\}$ of X_1-neighbourhoods of x such that, for each n, $V_{n+1} \subseteq V_n$ and $\bigcap_{n=1}^{\infty} V_n = x$. Given V_n in place of V above, let ϵ_n be the corresponding $\frac{1}{2}\delta$: we may clearly suppose that $\epsilon_n \to 0$. Let p_n, μ_n denote p_ϵ, μ_ϵ when $\epsilon = \epsilon_n$. Then all the μ_n have the end-points v_1, v_2, and $\mu_n \subseteq U(F, \epsilon_n)$. Hence, μ_n is deformable to μ_{n+1} in $U(F, \epsilon_n)$, so that $\theta_1 p_n \simeq \theta_1 p_{n+1} \operatorname{rel} x$ in V_n, say by a homotopy

$$\chi_n\colon (E^1 \times l, \dot{E}^1 \times l) \to (V_n, x).$$

Now define a homotopy

$$\chi\colon (E^1 \times l, \dot{E}^1 \times l) \to (X_1, x)$$

by $$\begin{cases} \chi(\xi, t) = \chi_n\left(\xi, \dfrac{t - n^{-1}}{n^{-1} - (n+1)^{-1}}\right) & \text{if } 1 - n^{-1} \leqslant t \leqslant 1 - (n+1)^{-1}, \\ \chi(\xi, 1) = x. \end{cases}$$

The continuity of χ at all (ξ, t) with $t < 1$ is immediate; but, given the neighbourhood V_n of x, then, for all t with $1 - n^{-1} < t \leqslant 1$, $\chi(\xi, t) \in V_n$. Hence χ is continuous at all points $(\xi, 1)$. Now, by (viii),

$$f_t \simeq \theta_1 p_1 \operatorname{rel} x \text{ in } X_1;$$

and $$\theta_1 p_1 = \chi_1 \mid (E^1 \times (0)) = \chi \mid (E^1 \times (0)),$$

so that $\theta_1 p_1 \simeq 0 \operatorname{rel} x$ in X_1. Therefore $f_t \simeq 0 \operatorname{rel} x$ in X_1, contrary to (iv). A similar contradiction is obtained if we start from a non-empty B_s distinct from all the other B_i. Hence the assertion of the theorem follows, and the proof is complete.

2.3. If Y is any path-wise connected space, then the singular integral homology group $\mathscr{H}^1(Y)$ is $\pi_1(Y)$ made Abelian. A proof of this assertion will be found in Hu (5) Theorem 12.5, and his proof is independent of any local properties in Y. Also, if G_r $(r = 1, 2)$ are groups and if G_r^α denotes G_r made Abelian, then $(G_1 \circ G_2)^\alpha$ is the direct sum $G_1^\alpha + G_2^\alpha$. In the notation of Theorem 1, let $j_r\colon \mathscr{H}^1(X_r) \to \mathscr{H}^1(X)$ be the injections $(r = 1, 2)$. Just as with the injections i_r in § 2.2 it follows that j_r is univalent. Moreover, if X_1, X_2 are each path-wise connected, then it is implicit in Hu's proof that

$$j_r \mathscr{H}^1(X_r) = (i_r \pi_1(X_r))^\alpha.$$

Hence we have the following corollaries to Theorems 3 and 1, respectively.

[2.4] COROLLARY. *If X_1, X_2 are path-wise connected, then the kernel of*

$$j_1 + j_2\colon \mathscr{H}^1(X_1) + \mathscr{H}^1(X_2) \to \mathscr{H}^1(X)$$

is zero.

[2.5] COROLLARY. *If, moreover, one of X_1, X_2 is $1-LC$ at x, then*

$$j_1 + j_2\colon \mathscr{H}^1(X_1) + \mathscr{H}^1(X_2) \approx \mathscr{H}^1(X).$$

2.6. It might be expected that, for Corollary 2.5, the '$1-LC$' condition in Theorem 1 could be weakened to the analogous condition '$1-lc_s$' using singular homology over the integers. But the composition of the homotopies ψ_{mn} to form ψ in (2.11) of the proof of Theorem 2 has no general analogue in singular homology (although it has in Vietoris's theory, for example); moreover, X_1 will be locally connected at x (in general) and no example seems to be known of such an X_1 which is $1-lc_s$ at x but not $1-LC$ at x. That some local condition is necessary for Corollary 2.5 to hold is shown at the end of the next section.

3. We shall require the following lemma.

3.1. LEMMA. *Let A be an arc-wise connected subset of a space X and let $\hat{A}$ denote the cone of base A. Let*

$$Y = X \cup \hat{A}, \qquad \hat{A} \cap X = A,$$

and let

$$j\colon \pi_1(A) \to \pi_1(X), \qquad k\colon \pi_1(X) \to \pi_1(Y)$$

denote the injections of the groups based at a fixed point $x \in A$. Then

$$\text{(i)} \quad k\pi_1(X) = \pi_1(Y),$$

and

$$\text{(ii)} \quad k^{-1}(0) = \{j\pi_1(A)\}^N,$$

where N denotes normal closure in $\pi_1(X)$.

Thus

$$\text{(iii)} \quad \pi_1(Y) \approx \pi_1(X)/\{j\pi_1(A)\}^N.$$

Proof. Let v be the vertex of $\hat{A}$. We first remark that, since $\hat{A}$ is locally contractible at v, then any loop in Y through v can obviously be deformed, in a neighbourhood of v, into a loop not through v. Let us consider $\hat{A}-v$ as the cylinder $A \times R_+$, where $R_+ = \langle 0, \infty \rangle \subseteq R^1$.

Thus, if $w \in \hat{A}-v$, then we can take w to be (z,t), for some unique $z \in A$, $t \in R_+$. This gives us a contraction $\theta: \hat{A}-v \to A$ defined by $\theta(w) = z$, whence we obtain a mapping

$$\phi: Y-v \to X$$

defined by $\phi(y) = \theta(y)$ if $y \in \hat{A}-v$, and $\phi(y) = y$ otherwise.

To prove (i) let $\alpha \in \pi_1(Y)$. Then, by the remark above, we may assume that α contains a loop g not through v. But then $g \simeq \phi g \operatorname{rel} x$ in $Y-v$, whence $\alpha = k\alpha'$, where α' is the element of $\pi_1(X)$ which contains ϕg. Hence k is 'onto', and (i) follows.

To prove (ii), let $\beta \in \pi_1(X)$, where $k\beta = 0$, and let $f \in \beta$ be a loop. Then $f \simeq 0 \operatorname{rel} x$ in Y. This homotopy can be represented by a mapping ψ of E^2 into Y such that $\psi \mid \dot{E}^2$ represents f and maps a fixed point $c \in \dot{E}^2$ onto x. If $v \notin \psi(E^2)$, then $\phi\psi$ gives a homotopy of f to zero in X, whence $\beta = 0$, i.e. $\beta \in \{j\pi_1(A)\}^N$. If $v \in \psi(E^2)$, let $F = \psi^{-1}(v)$; then $F \subseteq E^2-\dot{E}^2$, since $\psi(\dot{E}^2) \subseteq X$ and $v \notin X$.

Now, in E^2, F is closed, and disjoint from the closed set $\dot{E}^2$. Hence $\rho(F, \dot{E}^2) = \epsilon > 0$. Given $\delta > 0$, there is by Zoretti's theorem [see Whyburn (7) VI 3.11] a Jordan curve J in $U(F, \delta)$, such that F is contained in the interior G of J. Since ψ is continuous, we can take δ so small that $\delta < \epsilon$, whence $J \subseteq E^2-\dot{E}^2$, and $\psi(J) \subseteq \hat{A}$. Let q be the point of J nearest to c and let σ denote the oriented segment cq. Then, if J is oriented like $\dot{E}^2$, $\dot{E}^2$ is deformable to $\sigma . J . \sigma^{-1}$, in E^2-G. Therefore, if $\psi \mid J = h$ and $\psi(\sigma) = \lambda$, then $f \simeq \lambda . h . \lambda^{-1} \operatorname{rel} x$ in

$$\psi(E^2-G) \subseteq Y-v.$$

But then $f = \phi f \simeq \mu . \phi(h) . \mu^{-1} \operatorname{rel} x$ in X, where $\mu = \phi(\lambda)$. Let ν be a path joining $\phi\psi(q)$ to x in the arc-wise connected space A. Then $\mu . \nu$ is a loop in X, and belongs, say, to $\gamma \in \pi_1(X)$; while $\tau = \nu^{-1} . \phi(h) . \nu$ is a loop in A, and belongs, say, to $j\alpha \in \pi_1(x)$, where $\tau \in \alpha \in \pi_1(A)$. Therefore, rel x, we have

$$f \simeq \mu . \phi(h) . \mu^{-1} = (\mu . \nu) . (\nu^{-1} . \theta(h) . \nu) . (\mu . \nu)^{-1},$$

i.e. $\beta = \gamma\alpha\gamma^{-1}$, since $f \in \beta$. Hence, the kernel of k is contained in $\{j\pi_1(A)\}^N$; on the other hand, since $\hat{A}$ is contractible, $k\{j\pi_1(A)\}^N = 0$, and so the assertion (ii) follows. The assertion (iii) follows from Noether's theorem.

Remark. Since a closed two-cell is a cone, the proof needs only obvious modifications to show that, if we add a relation $\alpha = 0$ to $\pi_1(X)$, then the resulting group can be realized by attaching one two-cell to X, whereby $\dot{\sigma}$ is mapped into X by an element of α.

3.2. Let B be a second subset of X, and let $\hat{B} \cap Y = B$. Then, if

$$j' = \pi_1(B) \to \pi_1(X), \qquad l\colon \pi_1(X) \to \pi_1(Y \cup \hat{B})$$

are the injections, we have the diagram

$$\begin{array}{ccccc} \pi_1(A) & \overset{j}{\searrow} & & & \\ & & \pi_1(X) & \xrightarrow{k} & \pi_1(X \cup \hat{A}) \xrightarrow{l} \pi_1(X \cup \hat{A} \cup \hat{B}), \\ \pi_1(B) & \underset{j'}{\nearrow} & & & \end{array}$$

and as an extension of the previous lemma we have

[3.3] LEMMA. (i) $\pi_1(X \cup \hat{A} \cup \hat{B}) = lk\pi_1(X)$,

(ii) *the kernel of* lk *in* $\pi_1(X)$ *is*

$$L = \{j\pi_1(A).j'\pi_1(B)\}^N.\dagger$$

Proof. The assertion (i) follows immediately by applying § 3.1 (i). For the second, let the kernel of lk be K. Then, from the diagram, we obtain

$$\begin{aligned} lk(L) = lk\{j\pi_1(A).j'\pi_1(B)\}^N &= l(k\{j\pi_1(A)\}^N).(lk)(\{j'\pi_1(B)\}^N) \\ &= 0.0 = 0, \end{aligned}$$

using § 3.1 (ii) twice. On the other hand, if $\sigma \in K$, then $lk(\sigma) = 0$, i.e. $k(\sigma)$ is in the kernel of l. By § 3.1 (ii), this kernel is $\{kj'\pi_1(B)\}^\nu$, where ν denotes normal closure in $\pi_1(X \cup \hat{A})$. But k is 'onto', whence

$$\{kj'\pi_1(B)\}^\nu = k\{j'\pi_1(B)\}^N.$$

Therefore $$\sigma \in \{j'\pi_1(B)\}^N.\{j\pi_1(A)\}^N = L.$$

Hence $K \subseteq L \subseteq K$, and so $K = L$. This establishes § 3.3 (ii).

I shall now apply these results to a space C constructed as follows. Let $\{A_n\}$ be a sequence of spaces (assumed for simplicity to be metric), and let all the A_n have precisely one point x in common, being otherwise mutually disjoint.

Suppose that each A_n is $1-LC$ at x, and that

$$G_n = \pi_1(A_n, x) \neq 0.$$

Let $$A^{(1)} = \bigcup_{n=1}^{\infty} A_{2n-1}, \qquad A^{(2)} = \bigcup_{n=1}^{\infty} A_{2n}, \qquad A_* = A^{(1)} \cup A^{(2)},$$

with the topology in A_* such that

$$\lim_{n\to\infty} \operatorname{diam}(A_n) = 0.$$

† If G_1, G_2 are subgroups of a group, then $G_1.G_2$ denotes the set of all products $g_1 g_2$ $(g_i \in G_i)$.

and with each A_{2n-1} closed in $A^{(1)}$, each A_{2n} closed in $A^{(2)}$, and $A^{(r)}$ closed in A_* $(r = 1, 2)$. (This is easiest arranged if each X_n is compact.) Let C_r be the cone of base $A^{(r)}$ $(r = 1, 2)$ where $C_1 \cap C_2 = x$. Clearly, if $C = C_1 \cup C_2$, then C is not $1-LC$ at x. It will be our object to prove

[3.4] *The cardinal* $\|\pi_1(C, x)\|$ *is the cardinal* $\mathfrak{c}$ *of the continuum.*

Since C_1, C_2 are each contractible, then† $\pi_1(C_r) = 0$ $(r = 1, 2)$. Hence, if $i_r\colon \pi_1(C_r) \to \pi_1(C)$ are the injections, then

$$(i_1 \wedge i_2)(\pi_1(C_1) \circ \pi_1(C_2)) = 0,$$

so that, by virtue of [3.4], $i_1 \wedge i_2$ is very far from being 'onto'. It therefore remains to prove [3.4].

First of all, we have by [3.3] (iii) that

$$\begin{aligned} \pi_1(C) &\approx \pi_1(A_*)/\{i_1\pi_1(A^{(1)}) \cdot i_2\pi_1(A^{(2)})\}^N \\ &= \pi_1(A_*)/\{(i_1 \wedge i_2)(\pi_1(A^{(1)}) \circ \pi_1(A^{(2)}))\}^N \end{aligned} \tag{3.41}$$

by definition of $i_1 \wedge i_2$ and by Theorem 3. In order to discuss the groups in the right-hand side of (3.41), we have to introduce the 'topologist's product'

$$\mathop{\mathfrak{T}}_{n=1}^{\infty} G_n$$

of the groups G_n. This is defined‡ as follows.

3.5. For each m let

$$G^{(m)} = G_1 \circ G_2 \circ \ldots \circ G_m \subseteq G^{(m+1)}$$

and let $\theta_m\colon G^{(m+1)} \to G^{(m)}$ be the natural homomorphism defined by extending multiplicatively the correspondence

$$\theta_m(ab) = a \quad (a \in G^{(m)},\ b \in G_{m+1}).$$

Then we define the 'topologist's product' $\mathop{\mathfrak{T}}_{n=1}^{\infty} G_n$ of the G_n to consist of all sequences $\{g_m\}$, with $\{g_m\}\{h_m\} = \{g_m h_m\}$, where, for all m,

(i) $g_m \in G^{(m)}$ and $\theta_m g_{m+1} = g_m$

(ii) if g_m is of the form $x_{1m}x_{2m}\ldots x_{pm}$, where $0 \neq x_{im} \in G_{\sigma(i,m)}$, $\sigma(i,m) \neq \sigma(i+1,m)$ $(1 \leqslant i \leqslant p)$, then given r, there exists an integer $M = M(r)$ *independent of* m, such that the number of i with $\sigma(i,m) = r$ is less than M. (3.51)

Let G_* be the unrestricted free product of the G_n in the sense of Higman (**3**) and let

$$G^{\omega} = \bigcup_{m=1}^{\infty} G^{(m)}.$$

† We again omit the base point from the notations when no ambiguity arises.

‡ A study of this product will appear elsewhere, and will contain a proof of Theorem 4 below.

Then, as explained in Higman (**3**), there is an isomorphism ζ of G^ω onto a subgroup G_0^ω of G_*, and it is easily seen that†

$$\zeta G^\omega = G_0^\omega \subseteq \mathfrak{T} G_n \subseteq G_*. \tag{3.52}$$

An example of an element which is in G_* but not in $\mathfrak{T}G_n$ is $\{g_n\}$, where

$$g_n = [a_1, a_2][a_1, a_3]...[a_1, a_n], \quad 0 \neq a_i \in G_i.$$

We shall assume the following result‡

THEOREM 4. $$\pi_1(A_*, x) \approx \overset{\infty}{\underset{n=1}{\mathfrak{T}}} G_n.$$

Assuming this, we have, on writing A_* as the 'product' $(A_n)_*$,

$$A^{(1)} = (A_{2n-1})_*, \qquad A^{(2)} = (A_{2n})_*, \qquad A_* = (A_{2n-1} \cup A_{2n})_*,$$

so that $$\pi_1(A^{(1)}, x) \approx \mathfrak{T}G_{2n-1}, \qquad \pi_1(A^{(2)}, x) \approx \mathfrak{T}G_{2n}, \tag{3.53}$$

and $$\pi_1(A_*, x) \approx \mathfrak{T}\pi_1(A_{2n-1} \cup A_{2n}, x) \approx \mathfrak{T}(G_{2n-1} \circ G_{2n}), \tag{3.54}$$

by applying Theorem 1 to the spaces $A_{2n-1} \cup A_{2n}$. Write

$$\mathfrak{T}(G_{2n-1} \circ G_{2n}) = \Gamma \, (\approx \mathfrak{T}G_n).$$

Now let us recall (3.41). Since $i_1 \wedge i_2$ is univalent, then we obtain from (3.41), (3.53), and (3.54) that

$$\pi_1(C) \approx \Gamma / \{\mathfrak{T}G_{2n-1} \circ \mathfrak{T}G_{2n}\}^N, \tag{3.55}$$

where the N denotes normal closure in Γ. The assertion [3.4] is now an immediate consequence of the following purely algebraic theorem, to be proved in the next section.

THEOREM 5. *Let G be any normal subgroup of Γ satisfying*§

$$\mathfrak{T}G_{2n-1} \circ \mathfrak{T}G_{2n} \subseteq G \subseteq (\mathfrak{T}G_{2n-1} \circ \mathfrak{T}G_{2n}).[\Gamma, \Gamma],$$

where $[\Gamma, \Gamma]$ is the commutator subgroup of Γ. Then, if no G_n is trivial,

$$\|\Gamma / G\| \geqslant \mathfrak{c}.$$

By analogy with Wilder (**8**) [V 19.4, 167] let us call a space Z 'semi-1-connected in the homotopy sense' $(1-SLC)$ at $z \in Z$ if and only if there is a neighbourhood U of z such that the injection of $\pi_1(U, z)$ in $\pi_1(Z, z)$ is trivial. Thus $1-SLC$ is a weak form of $1-LC$ and both the cones C_1, C_2 are $1-SLC$ at x. This shows that the $1-LC$ condition in Theorem 1 cannot be relaxed much, if at all. Indeed, one is led to conjecture that the $1-LC$ condition at x is *equivalent* to the 'onto' property of $i_1 \wedge i_2$; but I have no proof to settle the matter.

† Where no ambiguity arises, we often omit the termini $n = 1$, ∞ from $\mathfrak{T}$. See second footnote on previous page.

‡ If H is any subgroup of a group with commutator K, then $H.K$ is normal because $xhkx^{-1} = [x, hk].hk \in K.(H.K) = H.K$.

§ See footnote † on p. 185.

Also, we note that the space C can be used to show that a local condition is necessary for Corollary (2.5) to hold. For, if $j_r\colon \mathscr{H}^1(C_r) \to \mathscr{H}^1(C)$ $(r = 1, 2)$ are the injections, then j_1+j_2 is 'onto' if and only if $\mathscr{H}^1(C) = 0$, because $\mathscr{H}^1(C_r) = 0$ and j_r has zero kernel $(r = 1, 2)$. But, since $\mathscr{H}^1(C)$ is $\pi_1(C, x)$ made Abelian, then by (3.55) we have

$$\mathscr{H}^1(C) \approx \Gamma/H.K,$$

where $H = \mathfrak{T}G_{2n-1} \circ \mathfrak{T}G_{2n}$ and $K = [\Gamma, \Gamma]$. But, by Theorem 5,

$$\|\Gamma/H.K\| \geqslant \mathfrak{c}.$$

Hence

$$\|\mathscr{H}^1(C)\| \geqslant \mathfrak{c}.$$

Therefore j_1+j_2 is not 'onto'. Admittedly neither C_1 nor C_2 is $1-lc_s$ [see § 2.6] at x. Note that every cycle of mesh $< \epsilon$ on C bounds a chain of mesh $< \delta(\epsilon)$ on C, where $\delta \to 0$ as $\epsilon \to 0$. Hence the integer Čech group $\mathscr{H}^1_c(C)$ is zero. This example should be compared with the well-known one of a space E based on the curve $\sin 1/x$, such that $\mathscr{H}^1_c(E) \neq 0$, yet $\mathscr{H}^1(E) = 0$.

4. This concluding section is devoted to the proof of Theorem 5, which runs as follows; $\{G_n\}$ and G^ω are as in (3.52).

In (**1**) I have introduced the functions $\beta_{pq}(x)$, which are defined (as we recall below) for every pair of distinct integers $p, q \geqslant 1$, and which assign an integer $\beta_{pq}(x)$ to every non-zero $x \in G^\omega$. Each such x is a unique product $x_1 x_2 \ldots x_r$, where $0 \neq x_i \in G_{\sigma(i)}$ and $\sigma(i) \neq \sigma(i+1)$ $(1 \leqslant i \leqslant r)$. We call $x_i x_{i+1}$ a '(p, q) pair in x' if and only if $x_i \in G_p$, $x_{i+1} \in G_q$. Let $\lambda_{pq}(x)$ be the number of (p, q) pairs in x; then $\beta_{pq}(x)$ is defined to be $\lambda_{pq}(x)-\lambda_{qp}(x)$. Hence, in particular, if

$$x \in G_1 \circ G_3 \circ \ldots \circ G_{2n+1} \quad \text{or} \quad x \in G_2 \circ G_4 \circ \ldots \circ G_{2n}$$

(for any n), then, for all $p > 0$,

$$\beta_{2p-1,2p}(x) = 0. \tag{4.1}$$

In (**1**) I proved that

$$\beta_{pq}(x) = -\beta_{pq}(x^{-1}), \tag{4.2}$$

and that

$$\beta_{pq}(x_1 x_2 \ldots x_s) = \sum_{i=1}^{s} \beta_{pq}(x_i) + \gamma_{pq}(x_1, x_2, \ldots, x_s), \tag{4.3}$$

where the $\gamma_{pq}(x_1, x_2, \ldots, x_s)$ are integers satisfying

$$\sum_{p,q} |\gamma_{pq}(x_1, x_2, \ldots, x_s)| \leqslant 6(s-1), \tag{4.4}$$

i.e. β_{pq} acts like a homomorphism on the product $x_1 x_2 \ldots x_s$ for all but at most $6(s-1)$ pairs (p, q).

As with the statement of Theorem 5, write

$$\Gamma = \mathfrak{T}G_n, \qquad H = \{\mathfrak{T}G_{2n} \circ \mathfrak{T}G_{2n-1}\}^N,$$

and put

$$K = [\Gamma, \Gamma].$$

It suffices to prove that $\|\Gamma/H.K\| \geqslant \mathfrak{c}$. Since each $G_n \neq 0$, we can choose $a_n \in G_n$, $a_n \neq 0$. Let $\mu = \{m(n)\}$ be a strictly increasing sequence of positive integers, and define

$$\alpha_n \in G^\omega \quad (n \geqslant 1)$$

by

$$\alpha_n = \begin{cases} a_{2n-1}a_{2n} & \text{if } n \in \mu, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\sigma_n = \alpha_1 \alpha_2 ... \alpha_n$, so that $\sigma_n \in G^\omega$ and $\sigma = \{\sigma_n\} \in \Gamma$. I assert that, if $\mu' = \{m'(n)\}$ is a second sequence like μ such that the set

$$M = M(\mu, \mu') = \{m(n)\} - \{m'(n)\}$$

is infinite, and if $\alpha_i', \sigma_n', \sigma'$ correspond to $\alpha_i, \sigma_n, \sigma$, then σ, σ' *are incongruent* (mod $H.K$).

For, if $\sigma^{-1}\sigma' \in H.K$, then $\sigma^{-1}\sigma'$ is a product of the form

$$g_1 h_1 ... g_r h_r k_1 ... k_s,$$

where

$$g_i \in \mathfrak{T}G_{2n-1}, \qquad h_i \in \mathfrak{T}G_{2n} \quad (1 \leqslant i \leqslant r),$$

and $k_1, ..., k_s$ are single commutators in K. Hence, for each n, we have (with $g_i = \{g_{in}\}$, etc.)

$$\sigma_n^{-1}\sigma_n' = g_{1n} h_{1n} ... g_{rn} h_{rn} k_{1n} ... k_{sn}, \tag{4.5}$$

while, by definition of σ_n, σ_n',

$$\sigma_n^{-1}\sigma_n' = \alpha_n^{-1}\alpha_{n-1}^{-1} ... \alpha_1^{-1}\alpha_1'\alpha_2' ... \alpha_n'. \tag{4.6}$$

If

$$k_{in} = u_{in} v_{in} u_{in}^{-1} v_{in}^{-1} \quad (i = 1, 2, ..., s),$$

then using the relations (4.1), (4.3), and the right-hand member of (4.5) we get, for all $p > 0$, that

$$\beta_{2p-1,2p}(\sigma_n^{-1}\sigma_n')$$

$$= \gamma_{2p-1,2p}(g_{1n}, h_{1n}, ..., g_{rn}, h_{rn}, u_{1n}, v_{1n}, u_{1n}^{-1}, v_{1n}^{-1}, ..., u_{sn}, v_{sn}, u_{sn}^{-1}, v_{sn}^{-1}).$$

Hence, by (4.4), the number of non-zero $\beta_{2p-1,2p}(\sigma_n^{-1}\sigma_n')$ is

$$\nu(\sigma_n^{-1}\sigma_n') \leqslant 6(r+4s-1). \tag{4.7}$$

On the other hand, with α_n as in the definition of σ_n, $\beta_{2p-1,2p}(\alpha_n)$ is 1 or 0; and, if the set M is arranged as a strictly increasing sequence $\lambda(m)$, then, even if cancellation is possible at $\alpha_1^{-1}\alpha_1'$ on the right of (4.6), it cannot continue beyond $\alpha_{\lambda(1)}^{-1}\alpha_{\lambda(1)}'$ (if $n > \lambda(1)$). For the first non-zero α_q'

with $q > \lambda(1)$ is such that $q \in \mu'$, and hence $q \notin M$; so that $\alpha_{\lambda(1)}, \alpha'_{\lambda(1)}$ are non-zero elements of different groups. Therefore, in particular,

$$\nu(\sigma^{-1}_{\lambda(n)}\sigma'_{\lambda(n)}) \geqslant n.$$

But, since M is infinite, we can take $n > 6(r+4s-1)$ in this last inequality, to contradict (4.7). This establishes the assertion that σ, σ' are incongruent (mod $H.K$).

It is shown in (**1**) that, if S is the set of all strictly increasing sequences μ of integers $\geqslant 1$ such that two distinct μ, $\mu' \in S$ have an infinite $M(\mu, \mu')$, then $\|S\| = \mathfrak{c}$. We have just shown that, if μ, $\mu' \in S$ ($\mu \neq \mu'$), then the corresponding σ's are in distinct elements of $\Gamma/H.K$. Hence

$$\|\Gamma/H.K\| \geqslant \mathfrak{c}$$

as required. This completes the proof of Theorem 5.

The proof applies equally well if we use the unrestricted free product (now denoted by $\mathfrak{H}$), so that we have also

THEOREM 6. *If U is the commutator subgroup of*

$$V = \mathop{\mathfrak{H}}_{n=1}^{\infty} G_n,$$

then $$\|V/(\{\mathfrak{H}G_{2n-1} \circ \mathfrak{H}G_{2n}\}.U)\| \geqslant \mathfrak{c},$$

provided that each G_n is non-trivial.

Finally, I wish to express my thanks to Dr. I. James for first bringing to my notice a space, of the form C in § 3, as an example of a space which is the union of two contractible spaces with only one common point, but which, by [3.4], is not itself contractible. I also wish to thank the referee for suggesting many improvements to the original draft of the paper and for pointing out an error.

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FOUR SQUARES AND A kth POWER

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1. LET $$f(x) = \sum_{r=1}^{n} \lambda_r x_r^2,$$

where λ_r are non-zero real numbers, not all of the same sign. It was shown by Chowla (**1**) that, if all ratios λ_r/λ_s $(r \neq s)$ are irrational and if $n \geqslant 9$, then $|f(x)|$ can be made arbitrarily small for integer values of $x_1,\ldots, x_n$, not all zero. This result was improved by Davenport and Heilbronn (**2**), who applied a modification of the Hardy–Littlewood method to replace 9 by 5 and the condition of irrationality by the weaker restriction that at least one λ_r/λ_s is irrational.

If the continued fraction for the irrational θ has bounded partial quotients, then $|x^2-\theta^2y^2|$ does not take small values for non-zero integers x, y. But it is not known whether or not the 5 in the Davenport–Heilbronn theorem can be replaced by 3 or 4. It seems to be worth recording that the result corresponding to that of Davenport–Heilbronn holds for

$$F(x) = \sum_{r=1}^{4} \lambda_r x_r^2 + \mu x_5^k,$$

where $\lambda_1,\ldots, \lambda_4, \mu$ are non-zero real numbers, not all of the same sign, at least one ratio λ_r/λ_s is irrational, and k is any positive integer. The proof is based on the analysis given by Watson (**4**), who applied the Davenport–Heilbronn method to prove a similar result for

$$\phi(x) = \sum_{r=1}^{5} \lambda_r x_r^2 + \lambda_6 x_4 x_5.$$

In fact the proof needs only slight modifications in Watson's work.

One way of interpreting the result would be to say that 5 in the Davenport–Heilbronn theorem can be replaced by $4+\epsilon$. It would tend to support the conjecture that 5 can be replaced at least by 4, if not 3.

Like Davenport–Heilbronn and Watson I prove more, namely

THEOREM. *Let $\lambda_1,\ldots, \lambda_4, \mu$ be non-zero real numbers, not all of the same sign and such that at least one ratio λ_r/λ_s is irrational. Suppose k a positive integer. Then there exist arbitrarily large P such that the inequalities*

$$1 \leqslant x_1,\ldots, x_4 \leqslant P^k, \qquad 1 \leqslant x_5 \leqslant P^2,$$

$$|F(x)| = \left|\sum_{r=1}^{4} \lambda_r x_r^2 + \mu x_5^k\right| < 1$$

have more than γP^{2k+2} solutions, where

$$\gamma = \gamma(\lambda_1,\ldots,\lambda_4,\mu) > 0.$$

Quart. J. Math. Oxford (2), 5 (1954), 191–202.

The result for $|F| < \epsilon$ follows by applying the above to F/ϵ. Without loss of generality we may suppose that $k \geqslant 2$ and that λ_1/λ_2 is irrational.

2. As far as possible I use the notation of (**2**) and (**4**). Small italic letters, other than e and i, denote integers and q, q_0,... denote positive integers. Small Greek letters denote real numbers; ϵ_0, ϵ_1, δ denote small positive numbers suitably fixed (in the context), while ϵ, wherever it appears, denotes a positive number which may be taken to be arbitrarily small; γ_1 denotes a positive number depending only on $\lambda_1,\ldots,\lambda_4, \mu$, while the constants implied by the symbol O depend at most on $\lambda_1,\ldots,\lambda_4, \mu$ and ϵ.

For any positive integer P we define

$$S(\alpha) = S(\alpha, P) = \sum_{x=1}^{P^k} \exp(2\pi i \alpha x^2) = \sum_{x=1}^{P^k} e(\alpha x^2),$$

$$K(\alpha) = K(\alpha, P) = \sum_{x=1}^{P^2} \exp(2\pi i \alpha x^k) = \sum_{x=1}^{P^2} e(\alpha x^k),$$

$$I(\alpha) = \int_0^{P^k} e(\alpha \xi^2)\, d\xi,$$

and

$$J(\alpha) = \int_0^{P^2} e(\alpha \xi^k)\, d\xi.$$

In all the lemmas quoted from (**2**) and (**4**) it is to be understood that P has been replaced by P^k.

LEMMA 1. [(**2**) Lemma 4]. *For any* η

$$\int_{-\infty}^{\infty} e(\eta\alpha)\left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha = \max(0, 1-|\eta|).$$

LEMMA 2.

$$\sum_{x_1=1}^{P^k} \ldots \sum_{x_4=1}^{P^k} \sum_{x_5=1}^{P^2} \max(0, 1-|F(x_1,\ldots,x_4,x_5)|)$$
$$= \int_{-\infty}^{\infty} S(\lambda_1\alpha)\ldots S(\lambda_4\alpha) K(\mu\alpha)\left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha,$$

$$\int_0^{P^k} \ldots \int_0^{P^k} \int_0^{P^2} \max(0, 1-|F(\xi_1,\ldots,\xi_4,\xi_5)|)\, d\xi_1 \ldots d\xi_4\, d\xi_5$$
$$= \int_{-\infty}^{\infty} I(\lambda_1\alpha)\ldots I(\lambda_4\alpha) J(\mu\alpha)\left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha.$$

The proof follows from Lemma 1.

LEMMA 3.

$$\int_0^{P^k}\cdots\int_0^{P^k}\int_0^{P^2}\max(0,1-|F(\xi_1,\ldots,\xi_4,\xi_5)|)\,d\xi_1\ldots d\xi_4 d\xi_5 > \gamma_1 P^{2k+2}.$$

This can be proved by a slight modification of the proof of (**2**) Lemma 10. In view of Lemmas 2 and 3 we have only to prove

LEMMA A_1.

$$\int_{-\infty}^{\infty} S(\lambda_1\alpha)\ldots S(\lambda_4\alpha)K(\mu\alpha)\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2 d\alpha$$

$$= \int_{-\infty}^{\infty} I(\lambda_1\alpha)\ldots I(\lambda_4\alpha)J(\mu\alpha)\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2 d\alpha + o(P^{2k+2}),$$

where P tends to ∞ along some sequence.

3. LEMMA d_1.

$$\text{(i)}\quad S(\alpha) = I(\alpha)+O\big(1+|\alpha|P^k+\alpha^2P^{3k}\big),$$

$$\text{(ii)}\quad K(\alpha) = J(\alpha)+O\big(1+|\alpha|P^{2k-2}+\alpha^2P^{4k-2}\big).$$

We prove (i) as in (**2**) Lemma 5, while (ii) follows in the same way from the Euler summation-formula.

LEMMA d_2. $\quad I(\alpha) = O\big(|\alpha|^{-\frac{1}{k}}\big),\ J(\alpha) = O\big(|\alpha|^{-1/k}\big).$

Proof.

$$J(\alpha) = \int_0^{P^2} e(\alpha\xi^k)\,d\xi = k^{-1}|\alpha|^{-1/k}\int_0^{|\alpha|P^{2k}} e(\pm\eta)\eta^{-(1-1/k)}\,d\eta$$

$$= O\big(|\alpha|^{-1/k}\big).$$

Similarly $\qquad I(\alpha) = O\big(|\alpha|^{-\frac{1}{k}}\big).$

LEMMA 4.

$$\int |I(\lambda_1\alpha)\ldots I(\lambda_4\alpha)J(\mu\alpha)|\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2 d\alpha = o(P^{2k+2}),$$

where the integral is taken over $|\alpha| \geqslant P^{-(2k-\epsilon_0)}$.

Proof. The left-hand member is

$$O\left(\int_{P^{-(2k-\epsilon_0)}}^{\infty}\alpha^{-2-1/k}\,d\alpha\right)$$

$$= O(P^{(1+1/k)(2k-\epsilon_0)})$$

$$= o(P^{2k+2}).$$

LEMMA 5.

$$\int \{S(\lambda_1\alpha)...S(\lambda_4\alpha)K(\mu\alpha)-I(\lambda_1\alpha)...I(\lambda_4\alpha)J(\mu\alpha)\}\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2 d\alpha = o(P^{2k+2}),$$

where the integral is taken over $-P^{-(2k-\epsilon_0)} \leqslant \alpha \leqslant P^{-(2k-\epsilon_0)}$.

Proof. By Lemma d_1, in the interval of integration, we have

$$S(\lambda_r\alpha)-I(\lambda_r\alpha) = O(1) \quad (r = 1,...,4),$$

$$K(\mu\alpha)-J(\mu\alpha) = O(1).$$

Since clearly $S(\lambda_r\alpha) = O(P^k)$ and $K(\mu\alpha) = O(P^2)$, it follows that

$$|S(\lambda_1\alpha)...S(\lambda_4\alpha)K(\mu\alpha)-I(\lambda_1\alpha)...I(\lambda_4\alpha)J(\mu\alpha)| = O(P^{4k}),$$

so that the left-hand member has the order

$$O\Big(\int P^{4k}\,d\alpha\Big) = O(P^{2k+\epsilon_0}) = o(P^{2k+2}),$$

where the integral is taken over $|\alpha| \leqslant P^{-(2k-\epsilon_0)}$.

Combining Lemmas A_1, 5, 4 we have only to show that, as $P \to \infty$ along a suitable sequence, we have

LEMMA A_2.

$$\int S(\lambda_1\alpha)...S(\lambda_4\alpha)K(\mu\alpha)\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2 d\alpha = o(P^{2k+2}),$$

where the integral is taken over $|\alpha| > P^{-(2k-\epsilon_0)}$.

4. LEMMA d_3 [(2) Lemma 6]. *If*

$$(a,q) = 1, \qquad \alpha = \frac{a}{q}+\beta, \qquad \beta = O(P^{-k}q^{-1}),$$

then

$$S(\alpha) = O(P^k q^{-\frac{1}{2}+\epsilon}+q^{\frac{1}{2}+\epsilon}).$$

LEMMA d_4 [(2) Lemma 7]. *If* $\alpha = O(P^{-k})$ *and* $\alpha \neq 0$, *then*

$$S(\alpha) = O\big(|\alpha|^{-\frac{1}{2}-\epsilon}\big).$$

LEMMA 6. *Let* $\sigma > 0$ *be any fixed constant. Then*

$$\int S(\lambda_1\alpha)...S(\lambda_4\alpha)K(\mu\alpha)\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2 d\alpha = o(P^{2k+2}),$$

where the integral is taken over $P^{-(2k-\epsilon_0)} \leqslant \alpha \leqslant \sigma P^{-k}$.

Proof. By Lemma d_4 in the interval of integration,

$$S(\lambda_r\alpha) = O\big(|\alpha|^{-\frac{1}{2}-\epsilon}\big).$$

Also, since $K(\mu\alpha) = O(P^2)$, the left-hand member has the order

$$O\left\{P^2 \int_{P^{-(2k-\epsilon_0)}}^{\infty} |\alpha|^{-2-4\epsilon}\, d\alpha\right\} = O(P^{2+(1+4\epsilon)(2k-\epsilon_0)}) = o(P^{2k+2})$$

since ϵ_0 is fixed and ϵ is arbitrarily small.

LEMMA d_5 [(**2**) Lemma 8].

(i) $$\int_m^{m+1} |S(\alpha)|^4\, d\alpha = O(P^{2k+\epsilon});$$

(ii) *when* $\rho \geqslant 0$ *and* $1 \leqslant r \leqslant 4$,

$$\int_{\rho}^{\infty} |S(\lambda_r \alpha)|^4 \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha = O\left(\frac{P^{2k+\epsilon}}{\rho+1}\right).$$

LEMMA 7.

$$\int_{|\alpha| \geqslant P^{\delta}} S(\lambda_1 \alpha)...S(\lambda_4 \alpha) K(\mu\alpha) \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha = o(P^{2k+2}),$$

where $\delta > 0$ *is any fixed number.*

This follows from Lemma d_5, since $K(\mu\alpha) = O(P^2)$ and

$$|S(\lambda_1 \alpha)...S(\lambda_4 \alpha)| \leqslant \tfrac{1}{4} \sum_{r=1}^{4} |S(\lambda_r \alpha)|^4.$$

Combining Lemmas A_2, 6, 7, we have only to prove that, as $P \to \infty$ through some sequence and for suitably fixed real positive numbers σ, δ, we have

$$\int S(\lambda_1 \alpha)...S(\lambda_4 \alpha) K(\mu\alpha) \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha = o(P^{2k+2}),$$

where the integral is taken over $\sigma P^{-k} < |\alpha| < P^{\delta}$.

In particular taking $\sigma = \lambda^{-1}$, where $\lambda = \frac{1}{2} \min_{1 \leqslant r \leqslant 4} |\lambda_r|$, we have only to prove

LEMMA A_3. *Let* $\lambda = \frac{1}{2} \min_{1 \leqslant r \leqslant 4} |\lambda_r|$. *Then*

$$\int S(\lambda_1 \alpha)...S(\lambda_4 \alpha) K(\mu\alpha) \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha = o(P^{2k+2}),$$

where the integral is taken over $\lambda^{-1} P^{-k} < |\alpha| < P^{\delta}$.

5. From now on the argument follows (**4**) very closely. For

$$\lambda P^{-k} < |\alpha| < P^{\delta}$$

we define, as in (**4**), the Farey fraction

$$\frac{a_r}{q_r} = \frac{a_r(\alpha)}{q_r(\alpha)} \quad (r = 1, 2, 3, 4)$$

as the last convergent with denominator not exceeding P^k to the continued fraction for $\lambda_r \alpha$. (When $\lambda_r \alpha$ is negative,

$$a_r = a(\lambda_r \alpha, P^k) = -a(|\lambda_r \alpha|, P^k), \qquad q_r = q(\lambda_r \alpha, P^k) = q(|\lambda_r \alpha|, P^k).$$

Then clearly

$$\lambda_r \alpha = \frac{a_r}{q_r} + \beta_r,$$

where

$$\beta_r \leqslant P^{-k} q_r^{-1},$$

so that the interval for α denoted by $M_r(a_r, q_r)$ $(r = 1, \ldots, 4)$ on which a_r, q_r have given values is of length $O(P^{-k} q_r^{-1})$.

Following Watson, using the estimate for $S(\alpha)$ given by Lemma d_3 and adding the contributions of different $M_r(a_r, q_r)$, we have

LEMMA W_1. [(**4**) Lemma 10].

$$\int |S(\lambda_r \alpha)|^4 \, d\alpha = o(P^{2k})$$

where the integral is taken over $\lambda^{-1} P^{-k} \leqslant |\alpha| \leqslant P^{-\delta}$.

LEMMA 8.

$$\int S(\lambda_1 \alpha) \ldots S(\lambda_4 \alpha) K(\mu\alpha) \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha = o(P^{2k+2})$$

where the integral is taken over $\lambda^{-1} P^{-k} \leqslant |\alpha| \leqslant P^{-\delta}$.

The proof follows from Lemma W_1 since $K(\mu\alpha) = O(P^2)$.

After Lemma 8 we need only prove that, as $P \to \infty$ through some sequence, we have

LEMMA A_4.

$$\int_{P^{-\delta} < |\alpha| < P^{\delta}} S(\lambda_1 \alpha) \ldots S(\lambda_4 \alpha) K(\mu\alpha) \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha = o(P^{2k+2}).$$

6. LEMMA W_2 [(4) (8.3) *and Lemma* 11 *with* q_4 *replaced by* q_2].

$$\text{(i)} \quad \int'_{m \leqslant \alpha \leqslant m+1} |S(\lambda_r \alpha)|^4 \, d\alpha = O(P^{2k+\epsilon-\delta}),$$

the accent denoting the omission of the part (possibly the whole) of every interval $M_r(a_r, q_r)$ *in which* $\beta_r < P^{\delta-2k}$ *in the interval of integration.*

(ii) *The above estimate holds for* $r = 1$ *if the accent is taken to denote the exclusion of intervals or parts thereof on which* $q_1 < P^{\delta} q_2$ *or* $|\beta_2| > P^{\delta-2k}$.

(iii) *The above holds if* 1 *and* 2 *are interchanged.*

LEMMA h. $$\int_m^{m+1} |K(\alpha)|^4 \, d\alpha = O(P^{4+\epsilon}).$$

The proof is the main lemma of Hua (**3**), with P^2 replacing p.

LEMMA d_6. *Suppose that for every* m

$$\int_m^{m+1} |G(\alpha)| \, d\alpha = O(P^l).$$

Then

$$\text{(i)} \quad \int_m^{m+1} |G(\lambda_r \alpha)| \, d\alpha = O(P^l),$$

$$\text{(ii)} \quad \int_{P^{-\delta} \leqslant |\alpha| \leqslant P^{\delta}} |G(\lambda_r \alpha)| \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha = O(P^l).$$

Proof. (i) $$\int_m^{m+1} |G(-\alpha)| \, d\alpha = \int_{-(m+1)}^{-m} |G(\alpha)| \, d\alpha = O(P^l).$$

Writing $\lambda_r = \lambda$ for convenience, we have

$$\begin{aligned} \int_m^{m+1} |G(\lambda\alpha)| \, d\alpha &= \frac{1}{|\lambda|} \int_{|\lambda| m}^{|\lambda|(m+1)} |G(\pm\alpha)| \, d\alpha \\ &= O\left\{ \int_{[|\lambda| m]}^{[|\lambda| m+1]} + \ldots \int_{[\lambda|(m+1)|]}^{[\lambda|(m+1)|]+1} \right\} |G(\pm\alpha)| \, d\alpha \\ &= O(P^l), \end{aligned}$$

where $[x]$ denotes the greatest integer not exceeding x.

$$\text{(ii)} \quad \int_{P^{-\delta}}^{P^{\delta}} |G(\lambda\alpha)| \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha \leqslant \int_{0}^{P^{\delta}} |G(\lambda\alpha)| \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha$$

$$\leqslant \int_{0}^{1} |G(\lambda\alpha)|\, d\alpha + \int_{1}^{\infty} |G(\lambda\alpha)| \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha$$

$$\leqslant \int_{0}^{1} |G(\lambda\alpha)|\, d\alpha + \sum_{m=1}^{\infty} \frac{1}{(\pi m)^2} \int_{m}^{m+1} |G(\lambda\alpha)|\, d\alpha$$

$$= O(P^l),$$

$$\int_{P^{-\delta} < -\alpha < P^{\delta}} |G(\lambda\alpha)| \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha = \int_{P^{-\delta} < \alpha < P^{\delta}} |G(-\lambda\alpha)| \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha = O(P^l).$$

The next lemma is similar to (4) Lemma 16.

LEMMA 9.

$$\int_{P^{-\delta} \leqslant |\alpha| \leqslant P^{\delta}}' S(\lambda_1\alpha)\ldots S(\lambda_4\alpha) K(\mu\alpha) \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha = o(P^{2k+2}),$$

where the accent denotes the omission of intervals on which α is such that all the following inequalities hold

$$\text{(i)} \quad |\beta_r| \leqslant P^{\delta-2k} \quad (r = 1,\ldots,4),$$

$$\text{(ii)} \quad q_1 < P^{k-2+\epsilon_1}, \quad q_2 < P^{k-2+\epsilon_1},$$

$$\text{(iii)} \quad q_1 < P^{\delta} q_2, \quad q_2 < P^{\delta} q_1.$$

Proof. Let $\int''$ denote the integration over only those α for which $|\beta_r| > P^{\delta-2k}$ for some r, say 1. Then

$$\int_{m}^{(m+1)}'' |S(\lambda_1\alpha)\ldots S(\lambda_4\alpha) K(\mu\alpha)|\, d\alpha$$

$$= O\left[P^2 \left\{ \int_{m}^{m+1} |S(\lambda_1\alpha)|^4\, d\alpha \int_{m}^{m+1} |S(\lambda_2\alpha)|^4\, d\alpha \ldots \int_{m}^{m+1} |S(\lambda_4\alpha)|^4\, d\alpha \right\}^{\frac{1}{4}}\right]$$

$$= O(P^2 (P^{2k+\epsilon-\delta} P^{6k+3\epsilon})^{\frac{1}{4}})$$

$$= o(P^{2k+2})$$

by Lemma W_2 (i), d_5 (i), and d_6 (i).

Therefore, by Lemma d_6 (ii), we have

$$\int_{P^{-\delta} \leqslant |\alpha| \leqslant P^{\delta}}'' |S(\lambda_1\alpha)\ldots S(\lambda_4\alpha) K(\mu\alpha)| \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2 d\alpha = o(P^{2k+2}). \qquad (1)$$

Suppose that $\int^* d\alpha$ denotes integration over α for which q_1 or q_2, say $q_1 \geqslant P^{k-2+\epsilon_1}$. Then, by Lemma d_3,

$$S(\lambda_1 \alpha) = O(P^{\frac{1}{2}k+1-\frac{1}{2}\epsilon_1+\epsilon(k-2+\epsilon_1)}) = O(P^{\frac{1}{2}k+1-\frac{1}{4}\epsilon_1}).$$

Then, applying the inequality of the arithmetic-geometric mean followed by that of Cauchy, we have

$$\begin{aligned}
&\int_m^{(m+1)*} |S(\lambda_1 \alpha)...S(\lambda_4 \alpha)K(\mu\alpha)|\, d\alpha \\
&\quad = O\Big(P^{\frac{1}{2}k+1-\frac{1}{4}\epsilon_1} \int_m^{m+1} |S(\lambda_2 \alpha)...S(\lambda_4 \alpha)K(\mu\alpha)|\, d\alpha\Big) \\
&\quad = O\Big(P^{\frac{1}{2}k+1-\frac{1}{4}\epsilon_1} \sum_{r=2}^{4} \Big\{\int_m^{m+1} |S(\lambda_r \alpha)|^4\, d\alpha\Big\}^{\frac{3}{4}} \Big\{\int_m^{m+1} |K(\mu\alpha)|^4\, d\alpha\Big\}^{\frac{1}{4}}\Big) \\
&\quad = O\{P^{\frac{1}{2}k+1-\frac{1}{4}\epsilon_1}(P^{2k+\epsilon})^{\frac{3}{4}}(P^{1+\epsilon})^{\frac{1}{4}}\} \\
&\quad = o(P^{2k+2})
\end{aligned}$$

by Lemmas d_3, h, and d_6.

Applying Lemma d_6 (ii) we get

$$\int_{P^{-\delta} \leqslant |\alpha| \leqslant P^{\delta}}^{*} |S(\lambda_1 \alpha)...S(\lambda_4 \alpha)K(\mu\alpha)| \Big(\frac{\sin \pi\alpha}{\pi\alpha}\Big)^2 d\alpha = o(P^{2k+2}). \tag{2}$$

Now suppose that $\int^\dagger d\alpha$ means integration over those α for which (i) holds and (iii) does not hold, i.e. either $q_1 \geqslant P^{\delta}q_2$ or $q_2 \geqslant P^{\delta}q_1$: suppose the former. Then, by Lemma W_2 (ii),

$$\int_{m \leqslant \alpha \leqslant m+1}^{\dagger} |S(\lambda_1 \alpha)|^4\, d\alpha = O(P^{2k+\epsilon-\delta})$$

and like (1) we derive

$$\int_{P^{-\delta} \leqslant |\alpha| \leqslant P^{\delta}}^{\dagger} |S(\lambda_1 \alpha)...S(\lambda_4 \alpha)K(\mu\alpha)| \Big(\frac{\sin \pi\alpha}{\pi\alpha}\Big)^2 d\alpha = o(P^{2k+2}), \tag{3}$$

and the lemma follows from (1), (2), and (3).

By Lemmas A_4 and 9 we have only to prove that, for a sequence of $P \to \infty$, we have

LEMMA A_5.

$$\int_{P^{-\delta} \leqslant |\alpha| \leqslant P^{\delta}}^{-} S(\lambda_1 \alpha)...S(\lambda_4 \alpha)K(\mu\alpha) \Big(\frac{\sin \pi\alpha}{\pi\alpha}\Big)^2 d\alpha = o(P^{2k+2}),$$

where the bar means that the integration is over those α for which

$$\text{(i)} \quad |\beta_r| \leqslant P^{\delta-2k} \quad (r = 1,\ldots,4),$$

$$\text{(ii)} \quad q_1 < P^{k-2+\epsilon_1}, \quad q_2 < P^{k-2+\epsilon_1},$$

$$\text{(iii)} \quad q_1 < P^{\delta} q_2, \quad q_2 < P^{\delta} q_1.$$

7. LEMMA W_3 [(4) Lemma 2]. *Let a/q be any convergent to the continued fraction for α. Then the inequality*

$$|\alpha - A/Q| < (4qQ)^{-1},$$

in which A, Q denote integers, not necessarily coprime, and Q is positive, is not solvable

(i) *for two different A and the same Q,*

(ii) *for any $Q < q$,*

or (iii) *for any two different values of Q differing by less than Q.*

Since λ_1/λ_2 is irrational, there exist infinitely many integers a_0, q_0 such that

$$q_0 > 0, \qquad (a_0, q_0) = 1, \qquad |\lambda_1/\lambda_2 - a_0/q_0| \leqslant q_0^{-2}.$$

We now restrict our P to those of the form $P = [q_0^{1/(4-4\delta)}]$, where $[x]$ denotes the greatest integer not exceeding x. Clearly

$$q_0 = O(P^{4-4\delta}).$$

Throughout the rest of the paper we assume conditions (i), (ii), and (iii) of Lemma A_5 to be satisfied.

Since

$$|\beta_r| = \left|\lambda_r \alpha - \frac{a_r}{q_r}\right| = O(P^{\delta-2k}) \quad \text{and} \quad P^{-\delta} \leqslant |\alpha| \leqslant P^{\delta},$$

it follows that $\qquad a_r = O(P^{\delta} q_r), \qquad q_r = O(P^{\delta} a_r).$

Writing $A = a_1 q_2$, $Q = a_2 q_1$, we have

$$q_1^{-1} = \left(\frac{1}{a_2 q_1} \cdot \frac{a_2}{q_2} \cdot \frac{q_2}{q_1}\right)^{\frac{1}{2}} = O(Q^{-\frac{1}{2}} P^{\delta}), \quad q_2^{-1} = \left(\frac{1}{a_2 q_1} \cdot \frac{q_1}{q_2} \cdot \frac{a_2}{q_2}\right)^{\frac{1}{2}} = O(Q^{-\frac{1}{2}} P^{\delta}).$$

Therefore, by Lemma d_3,

$$S^4(\lambda_1 \alpha) = O(P^{4k+3\delta} Q^{-1}), \qquad S^4(\lambda_2 \alpha) = O(P^{4k+3\delta} Q^{-1}). \tag{4}$$

Also, since

$$\left|\lambda_1 \alpha - \frac{a_2}{q_1}\right| = O(P^{\delta-2k}), \qquad \left|\lambda_2 \alpha - \frac{a_2}{q_2}\right| = O(P^{\delta-2k}),$$

$$\alpha = O(P^{\delta}), \qquad \alpha^{-1} = O(P^{\delta}),$$

we have

$$\left|\frac{\lambda_1}{\lambda_2}-\frac{A}{Q}\right| = \left|\frac{\lambda_1}{\lambda_2}-\frac{a_1q_2}{a_2q_1}\right| = \left|\frac{a_2}{q_2\lambda_2}-\frac{a_1}{q_1\lambda_1}\right|\frac{\lambda_1q_2}{a_2}$$

$$= \frac{\lambda_1q_2}{a_2}\left|\frac{a_2}{q_2\lambda_2}-\alpha+\alpha-\frac{a_1}{q_1\lambda_1}\right|$$

$$\leqslant \frac{\lambda_1q_2}{a_2}\left(\frac{1}{\lambda_2}\left|\frac{a_2}{q_2}-\alpha\lambda_2\right|+\frac{1}{\lambda_1}\left|\lambda_1\alpha-\frac{a_1}{q_1}\right|\right)$$

$$= O(P^{\delta}P^{\delta-2k}) = O(P^{2\delta-2k}). \tag{5}$$

LEMMA W_4. *Under the conditions of Lemma* A_4 *and for* P *as chosen above*

$$\int |S(\lambda_1\alpha)S(\lambda_2\alpha)|^2\,d\alpha = O(P^{2k-4+9\delta}).$$

Proof. Since

$$a_2 = O(P^{\delta}q_2), \qquad q_1 = O(P^{k-2+\epsilon_1}), \qquad q_2 = O(P^{k-2+\epsilon_1}),$$

$$q_0 = O(P^{4-4\delta}),$$

we have

$$q_0Q = q_0a_2q_1 = O(P^{4-4\delta+\delta+2k-4+2\epsilon_1}) = O(P^{2k-3\delta+2\epsilon_1}) = o(P^{2k-2\delta})$$

(because δ and ϵ_1, on which there has been no restriction so far, could be chosen to have $\delta > 2\epsilon_1$). Hence $(q_0Q)^{-1} > \tau(P^{2\delta-2k})$ for any constant τ and P large enough. Hence by (2) and Lemma W_3 for P large enough the inequalities

$$Rq_0 \leqslant Q < (R+1)q_0$$

have no solutions for $R = 0$, and one at most for each of $R = 1, 2, 3,\ldots$, with a unique A for each Q. Clearly A and Q are non-zero integers bounded by a power of P and so factorize in $O(P^{\epsilon})$ ways to give $O(P^{\epsilon})$ sets of values of a_1, q_1, a_2, q_2. Each of the $O(P^{\epsilon}) = O(P^{\delta})$ intervals so determined on which all the conditions of the lemma hold is of length $O(P^{\delta-2k})$ and on all of them, by (1),

$$|S(\lambda_r\alpha)|^4 = O(P^{4k+3\delta}Q^{-1}) \quad (r = 1, 2).$$

Hence each R contributes to the integral of the lemma at most

$$O(P^{2k+4\delta}Q^{-1}) = O(P^{2k+4\delta}R^{-1}q_0^{-1}) = O(P^{2k+8\delta-4}R^{-1}).$$

The lemma follows on summing over $R = 1, 2, 3,\ldots$ since R is bounded by a power of P.

Proof of Lemma A_5. By using the facts

$$\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2 \leqslant 1, \qquad K(\mu\alpha) = O(P^2),$$

together with Cauchy's inequality and Lemmas W_3 and d_5, we get

$$\int^{-} |S(\lambda_1\alpha)\dots S(\lambda_4\alpha)K(\mu\alpha)|\left(\frac{\sin\pi\alpha}{\pi\alpha}\right)^2 d\alpha$$

$$= O\Big[P^2\Big\{\int^{-} S(\lambda_1\alpha)S(\lambda_2\alpha)|^2\,d\alpha\Big\}^{\frac{1}{2}}\times$$

$$\times\Big\{\int_0^{P^\delta}|S(\lambda_3\alpha)|^4\,d\alpha\Big\}^{\frac{1}{4}}\Big\{\int_0^{P^\delta}|S(\lambda_4\alpha)|^4\,d\alpha\Big\}\Big]^{\frac{1}{4}}$$

$$= O\{P^2P^{k-2+(9/2)\delta}(P^\delta P^{2k+\epsilon})^{\frac{1}{2}}\}$$

$$= o(P^{2k+2}).$$

With the proof of Lemma A_5 the theorem is completely proved.

In conclusion I would like to express my gratitude to Professor Atle Selberg for many useful discussions.

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ON THE PRIMITIVE LATTICE POINTS IN THE REGION $|(x+c)y| \leqslant 1$

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1. LET $L_1 = \alpha x+\beta y$, $L_2 = \gamma x+\delta y$ be two real homogeneous linear forms in the variables x, y, with determinant $\Delta = \alpha\delta-\beta\gamma \neq 0$; and let $c \neq 0$ be any constant. The purpose of this note is to establish the following result.

THEOREM 1. *There exist coprime integers x, y satisfying the inequality*

$$|(L_1+c)L_2| \leqslant (1-\tfrac{1}{2}\sqrt{2})|\Delta|. \tag{1}$$

The sign of equality in (1) *is necessary if and only if*

$$L_1 = \lambda_1 y, \qquad L_2 = \lambda_2(x+(\sqrt{2}-1)y), \qquad c = -(1-\tfrac{1}{2}\sqrt{2})\lambda_1, \tag{2}$$

apart from a homogeneous integral unimodular substitution on the variables x, y.

In an earlier paper (**1**) I investigated this problem by employing a method due to D. B. Sawyer (**5**); and the argument led to a stronger inequality, with $1-\frac{1}{2}\sqrt{2}$ replaced by $\frac{1}{4}$, when α/β is irrational. However, as Dr. Cassels has pointed out to me, this method contains an unjustified assumption. As far as concerns Sawyer's application of the method, the assumption can be completely justified [see Mordell (**4**)]; but, as I shall prove in this note, not only is my earlier proof of the inequality (1) defective, but the stronger inequality asserted above when α/β is irrational does not hold.† I shall show that (1) is the best possible inequality whether α/β is irrational or not. This follows from

THEOREM 2. *Let δ be any positive number, and let P_n, Q_n be positive integers such that $P_n/Q_n \to 2+\sqrt{2}$ as $n \to \infty$. Let*

$$L_1+c = x+(2P_n+\sqrt{2})y-2Q_n,$$
$$L_2 = x-(\sqrt{2}-1)y,$$
$$\Delta_n = 2P_n+2\sqrt{2}-1.$$

Then, provided that n is sufficiently large, the inequality

$$|(L_1+c)L_2| < (1-\tfrac{1}{2}\sqrt{2}-\delta)\Delta_n \tag{3}$$

has no solution in coprime integers x, y.

† As a consequence, the inequality (4) in the statement of my theorem on bilinear forms [(**1**) 119] should be replaced by

$$|B(x, y; z, t)| \leqslant (\tfrac{1}{2}-\tfrac{1}{4}\sqrt{2})|\Delta|.$$

Apart from this, the theorem stands.

Quart. J. Math. Oxford (2), 5 (1954), 203–11.

The conclusions of Theorems 1 and 2 taken together form a contrast with those established by Davenport (**2**) for an analogous problem. We observe that the inequality (1) holds when the condition that x, y be relatively prime is replaced by the weaker condition that x, y be not both zero. Furthermore, it is the best possible inequality; for, as was proved in (**1**), we have

$$|(y-(1-\tfrac{1}{2}\sqrt{2}))(x+(\sqrt{2}-1)y)| \geqslant 1-\tfrac{1}{2}\sqrt{2} \tag{4}$$

for all integers x, y, not both zero. Now, Davenport has shown that, when α/β is irrational, the stronger inequality

$$|(L_1+c)L_2| < \frac{1}{4\cdot 1}|\Delta| \tag{5}$$

can be satisfied by integers x, y, not both zero. This was improved by P. Kanagasabapathy (**3**), who obtained the constant $(4\cdot 25)^{-1}$ in place of $(4\cdot 1)^{-1}$; but the exact constant has not been determined. Thus, in essence, there is an 'isolation result' in Davenport's problem, but not in mine.

2. The following lemma due to Davenport (**2**), which I quote without proof, gives a convenient reduction for the forms under consideration.

LEMMA 1. *Let α/β and γ/δ be irrational numbers. Then there exists a homogeneous integral unimodular substitution on the variables x, y which transforms $(L_1+c)L_2$ into*

$$\pm\frac{\Delta}{\theta+\phi}(x+\theta y-\psi)(x-\phi y), \tag{6}$$

where

$$\theta > 1, \qquad 0 < \phi < 1, \qquad 1 \leqslant \psi < \theta. \tag{7}$$

We shall require some result for the case when α/β is rational, but as a precisely similar reduction is not available in all cases, I give the following partial analogue:

LEMMA 2. *Suppose that α/β is rational and that γ/δ is irrational. Then there exists a homogeneous integral unimodular substitution on the variables x, y which transforms $(L_1+c)L_2$ into*

$$\pm\frac{\Delta}{1+\phi}(x+y-\psi)(x-\phi y), \tag{8}$$

where $0 < \phi < 1$.

We shall need to extend this reduction when $|\psi| > 1$, and so it is convenient to consider two cases according as $|\psi| > 1$ or $|\psi| \leqslant 1$.

LEMMA 3. *If $|\psi| > 1$ and ϕ is an irrational satisfying $0 < \phi < 1$, the form*

$$\frac{\Delta}{1+\phi}(x+y-\psi)(x-\phi y)$$

can be transformed by a homogeneous integral unimodular substitution into

$$\pm\frac{\Delta}{\theta'+\phi'}(x+\theta' y-\psi')(x-\phi' y), \tag{9}$$

where $$\theta' > 1, \qquad 0 < \phi' < 1, \qquad 1 \leqslant \psi' < \theta'. \tag{10}$$

Clearly, if the hypotheses of Lemmas 2 and 3 hold, the form obtained in Lemma 3 is precisely analogous to that in Davenport's lemma. In the remaining case, we do not continue the reduction but establish the main result directly.

LEMMA 4. *Suppose that $|\psi| \leqslant 1$, $0 < \phi < 1$. Then there are coprime integers x, y satisfying*

$$\frac{1}{1+\phi}|(x+y-\psi)(x-\phi y)| \leqslant 1-\tfrac{1}{2}\sqrt{2}, \tag{11}$$

with strict inequality unless $\phi = 1/\sqrt{2}$, $\psi = (2+\sqrt{2})^{-1}$.

Lemmas 2, 3, and 4 are proved in § 3, and Theorem 1 is deduced from them in § 4.

3. In the proofs of Lemmas 2 and 3, it is clear that, without any loss of generality, we can assume that $|\Delta| = 1$.

Proof of Lemma 2. By interchanging x and y, if necessary, we can write

$$L_1+c = \lambda_1(x+\theta y-\psi), \qquad L_2 = \lambda_2(x-\phi y),$$

where $\lambda_1\lambda_2(\theta+\phi) = \pm 1$, θ is rational and ϕ is irrational. Put $\theta = p/q$, where p, q are coprime integers, and set

$$x' = qx+py.$$

Then there are integers r, s such that $pr-qs = \pm 1$, and we can put

$$y' = rx+sy.$$

Hence, in the new variables,

$$L_1+c = \lambda_1'(x'-\psi'), \qquad L_2 = \lambda_2'(x'-\phi' y').$$

We now apply a further unimodular substitution of the form

$$x' = x''+y'', \qquad y' = mx''+(m+1)y'',$$

where m is an integer. Thus, we have

$$L_1+c = \lambda_1'(x''+y''-\psi').$$
$$L_2 = \lambda_2'[(1-m\phi')x''+\{1-(m+1)\phi'\}y''].$$

We now fix the integer m by choosing it so that

$$\textit{either} \qquad -1 < \frac{1-(m+1)\phi'}{1-m\phi'} < 0,$$

$$\textit{or} \qquad -1 < \frac{1-m\phi'}{1-(m+1)\phi'} < 0.$$

In fact, we take $$m = [\phi'^{-1}],$$

since then $$\frac{\phi'^{-1}-(m+1)}{\phi'^{-1}-m} < 0,$$

and now one of the above alternatives must hold. Then, by interchanging x'' and y'', if necessary, we can ensure that

$$L_1+c = \lambda_1''(x''+y''-\psi''), \qquad L_2 = \lambda_2''(x''-\phi''y''),$$

where $0 < \phi'' < 1$. By comparison of determinants,

$$\lambda_1''\lambda_2''(1+\phi'') = \pm 1.$$

Hence, by a homogeneous integral unimodular substitution on the variables x, y, $(L_1+c)L_2$ is equivalent to one of the forms

$$\pm\frac{1}{1+\phi''}(x+y-\psi'')(x-\phi''y),$$

where $0 < \phi'' < 1$. This proves the lemma.

Proof of Lemma 3. We write†

$$L_{10}+c_0 = \lambda_0(x_0+\theta_0 y_0-\psi_0), \qquad L_{20} = \mu_0(x_0-\phi_0 y_0),$$

where

$$\lambda_0\mu_0 = \pm(1+\phi_0)^{-1}, \qquad \theta_0 = 1, \qquad \psi_0 = \psi, \qquad \phi_0 = \phi, \qquad |\psi_0| > 1.$$

Let the continued-fraction development of ϕ_0 be, say,

$$(0, a_{-1}, a_{-2}, a_{-3}, \ldots).$$

We can also put, formally,

$$\theta_0 = 1 = (a_0).$$

Then we define numbers θ_n, ϕ_n for every integer n ($\leqslant 0$) by the relations

$$\theta_n = (a_n, a_{n+1}, \ldots, a_0),$$

$$\phi_n = (0, a_{n-1}, a_{n-2}, \ldots),$$

and define a chain of integral unimodular substitutions

$$\left.\begin{aligned} x_n &= -a_n x_{n+1}+y_{n+1} \\ y_n &= x_{n+1} \end{aligned}\right\} \quad (n \leqslant -1).$$

† I follow the lines of Davenport's work; see (**2**), Lemma 2.

In this way we derive a chain of pairs of forms equivalent to $L_{10}+c_0$, L_{20}, namely

$$L_{1n}+c_n = \lambda_n(x_n+\theta_n y_n-\psi_n), \qquad L_{2n} = \mu_n(x_n-\phi_n y_n) \quad (n \leqslant 0).$$

This is easily verified, by induction, if we use the relations

$$\begin{aligned} x_n+\theta_n y_n &= -a_n x_{n+1}+y_{n+1}+\theta_n x_{n+1} = \theta_{n+1}^{-1}x_{n+1}+y_{n+1} \\ &= \theta_{n+1}^{-1}(x_{n+1}+\theta_{n+1}y_{n+1}), \\ x_n-\phi_n y_n &= -a_n x_{n+1}+y_{n+1}-\phi_n x_{n+1} = -\phi_{n+1}^{-1}x_{n+1}+y_{n+1} \\ &= -\phi_{n+1}^{-1}(x_{n+1}-\phi_{n+1}y_{n+1}). \end{aligned}$$

From these, we deduce that

$$\lambda_n = \theta_{n+1}\lambda_{n+1}, \qquad \mu_n = -\phi_{n+1}\mu_{n+1},$$

$$\lambda_n\mu_n = \pm(\theta_n+\phi_n)^{-1},$$

$$c_0 = -\lambda_n\phi_n.$$

Hence $$\frac{\psi_{n+1}}{\psi_n} = \frac{\lambda_n}{\lambda_{n+1}} = \theta_{n+1},$$

i.e. $$\frac{\psi_0}{\psi_{-1}} = \theta_0 = 1, \qquad \frac{\psi_{-1}}{\psi_{-2}} = \theta_{-1} > 1, \qquad \frac{\psi_{-2}}{\psi_{-3}} = \theta_{-2} > 1, \quad \ldots.$$

Also, we have $\theta_n\theta_{n+1} > 2$ for every $n \leqslant -2$, and so

$$|\psi_n| \to 0 \quad \text{as } n \to \infty.$$

Now $|\psi_0| > 1$ by hypothesis. Since also $\psi_{-1} = \psi_0$, we can choose an integer $n < 0$ such that

$$|\psi_{n-1}| < 1 \leqslant |\psi_n|,$$

which implies that $$1 \leqslant |\psi_n| < \theta_n.$$

Finally, we observe that, if $\psi_n < 0$, we can change the sign of both variables, since this has the effect of changing the sign of ψ_n only. This completes the proof.

Proof of Lemma 4. By changing (x, y) into $(-x, -y)$, if necessary, we can suppose that

$$0 \leqslant \psi \leqslant 1.$$

Then, writing $k = 2+\sqrt{2}$ for convenience, we shall assume that

$$(1-\psi)\phi \geqslant (1+\phi)k^{-1}, \tag{12}$$

$$(2-\psi)(1-\phi) \geqslant (1+\phi)k^{-1}, \tag{13}$$

$$\psi(1+\phi) = (1+\phi)k^{-1}, \tag{14}$$

and deduce a contradiction unless $\phi = 2^{-\frac{1}{2}}$, $\psi = k^{-1}$. From (12), (13), (14), we have

$$\begin{aligned} 1 &= \frac{2\phi}{1+\phi} + \frac{1-\phi}{1+\phi} \\ &\geqslant k^{-1}\left(\frac{2}{1-\psi} + \frac{1}{2-\psi}\right) \\ &\geqslant k^{-1}\left(\frac{2}{1-k^{-1}} + \frac{1}{2-k^{-1}}\right) \\ &= \frac{2}{k-1} + \frac{1}{2k-1} \\ &= 1, \end{aligned}$$

with strict inequality unless

$$\psi = k^{-1}, \qquad (1-\psi)\phi = (1+\phi)k^{-1}, \qquad (2-\psi)(1-\phi) = (1+\phi)k^{-1},$$

i.e.

$$\psi = k^{-1}, \qquad \phi = 2^{-\frac{1}{2}}.$$

Thus we have a contradiction, except in this special case. This proves the lemma.

4. *Proof of Theorem* 1. Clearly, if γ/δ is rational, the inequality (1) can be satisfied trivially with

$$L_2 = 0, \qquad (x, y) = 1.$$

Thus we may suppose henceforward that γ/δ is irrational and, without loss of generality, that $|\Delta| = 1$. The reduction developed in Lemmas 1, 2, and 3 can now be applied. In this way, *either* we obtain a product

$$(\theta+\phi)^{-1}(x+\theta y-\psi)(x-\phi y),$$

with $\theta > 1$, $0 < \phi < 1$, $1 \leqslant \psi < \theta$, equivalent to $\pm(L_1+c)L_2$, *or* we obtain a product

$$(1+\phi)^{-1}(x+y-\psi)(x-\phi y),$$

with $|\psi| \leqslant 1$, $0 < \phi < 1$, again equivalent to $\pm(L_1+c)L_2$. In the latter case, Lemma 4 applies, and we see that this product satisfies (1) with strict inequality, unless $\phi = 1/\sqrt{2}$, $\psi = (2+\sqrt{2})^{-1}$; when it remains to observe that the products

$$\begin{aligned} &\frac{\sqrt{2}}{1+\sqrt{2}}\{x+y-(1-\tfrac{1}{2}\sqrt{2})\}(x-\tfrac{1}{2}\sqrt{2}y), \\ &-\{y'-(1-\tfrac{1}{2}\sqrt{2})\}\{x'+(\sqrt{2}-1)y'\} \end{aligned} \tag{15}$$

are equivalent under the substitution $(x, y) \to (-x', x'+y')$.

To prove the inequality (1) in the first case, we shall assume that

$$|(x+\theta y-\psi)(x-\phi y)| \geqslant k^{-1}(\theta+\phi),$$

where $\theta > 1$, $0 < \phi < 1$, $1 \leqslant \psi < \theta$, and $k = 2+\sqrt{2}$, for four special sets of coprime integers x, y: namely,

$$(\psi-1) \geqslant k^{-1}(\theta+\phi), \tag{16}$$

$$(\theta-\psi)\phi \geqslant k^{-1}(\theta+\phi), \tag{17}$$

$$(1+2\theta-\psi)|1-2\phi| \geqslant k^{-1}(\theta+\phi), \tag{18}$$

$$(1+\theta-\psi)(1-\phi) \geqslant k^{-1}(\theta+\phi), \tag{19}$$

and deduce a contradiction. Firstly, if $\phi < \frac{1}{2}$, we use (17) and (18) to obtain

$$1 \geqslant \frac{2(\theta+\phi)}{k\theta-k\psi} + \frac{\theta+\phi}{k+2k\theta-k\psi},$$

while, if $\phi > \frac{1}{2}$, we use (18) and (19) to obtain

$$1 \geqslant \frac{2(\theta+\phi)}{k+k\theta-k\psi} + \frac{\theta+\phi}{k+2k\theta-k\psi}.$$

But, on using the inequality for $k\psi$ in (16), we have

$$0 < k\theta-k\psi < k+k\theta-k\psi \leqslant k\theta-(\theta+\phi) < (k-1)\theta,$$

$$0 < k+2k\theta-k\psi \leqslant k+2k\theta-k-(\theta+\phi) < (2k-1)\theta.$$

Hence, in both cases, we deduce that

$$\begin{aligned} 1 &> \frac{2(\theta+\phi)}{(k-1)\theta} + \frac{\theta+\phi}{(2k-1)\theta} \\ &> \frac{2\theta}{(k-1)\theta} + \frac{\theta}{(2k-1)\theta} \\ &= \frac{2}{k-1} + \frac{1}{2k-1} \\ &= 1; \end{aligned}$$

and this is a contradiction. Thus, the inequality (1) is completely established. The second part of the theorem now follows from the fact that the particular product (15) has absolute value at least $1-\frac{1}{2}\sqrt{2}$ for all integers x, y not both zero.

5. *Proof of Theorem* 2. We suppose that there is a pair of coprime integers x, y satisfying (3), and deduce a contradiction when n is sufficiently large. Note that

$$|(L_1+c)-L_2| = |\Delta_n y - 2Q_n| \geqslant 2Q_n \tag{20}$$

since the expression in y is obviously least when $y = 0$. Hence

$$|L_1+c|+|L_2| \geqslant 2Q_n \sim (1-\tfrac{1}{2}\sqrt{2})\Delta_n \quad \text{as } n \to \infty. \tag{21}$$

Now, if n is sufficiently large, the inequalities (3) and (21) imply that

$$\textit{either} \quad |L_1+c| < 1 \quad \textit{or} \quad |L_2| < 1.$$

If $|L_1+c| < 1$, we write

$$x = -(2P_n+\sqrt{2})y+2Q_n+\xi, \quad \text{where } |\xi| < 1.$$

Let $$(L_1+c)(L_1'+c') = (x+2P_n y-2Q_n)^2-2y^2 = N.$$

Then $|N| \geqslant 1$, since x, y are coprime. Hence, with $|y| \geqslant 1$ and n large,

$$\begin{aligned}|(L_1+c)L_2| \geqslant \left|\frac{L_2}{L_1'+c'}\right| &= \left|\frac{-(2P_n+\sqrt{2})y+2Q_n+\xi-(\sqrt{2}-1)y}{-(2P_n+\sqrt{2})y+2Q_n+\xi+(2P_n-\sqrt{2})y-2Q_n}\right| \\ &= \left|\frac{(2P_n+2\sqrt{2}-1)y_n-2Q_n-\xi}{2\sqrt{2}y-\xi}\right| \\ &= \frac{1}{2\sqrt{2}}\left|\frac{y-(2Q_n+\xi)/\Delta_n}{y-\xi/2\sqrt{2}}\right|\Delta_n \\ &\geqslant \frac{1}{2\sqrt{2}}\left(\frac{|y|-A_n}{|y|+\frac{1}{4}\sqrt{2}}\right)\Delta_n,\end{aligned}$$

where $$A_n = 2Q_n/\Delta_n+1/\Delta_n \to 1-\tfrac{1}{2}\sqrt{2} \quad \text{as } n \to \infty.$$

Similarly, if $|L_2| < 1$, we write

$$x = (\sqrt{2}-1)y+\eta, \quad \text{where } |\eta| < 1.$$

Let $$L_2 L_2' = (x+y)^2-2y^2 = N'.$$

Then $|N'| \geqslant 1$ since x, y are coprime. Hence, with $|y| \geqslant 1$ and n large,

$$\begin{aligned}|(L_1+c)L_2| \geqslant \left|\frac{L_1+c}{L_2'}\right| &= \left|\frac{(\sqrt{2}-1)y+\eta+(2P_n+\sqrt{2})y-2Q_n}{(\sqrt{2}-1)y+\eta+(\sqrt{2}+1)y}\right| \\ &= \frac{1}{2\sqrt{2}}\left|\frac{y-(2Q_n-\eta)/\Delta_n}{y+\eta/2\sqrt{2}}\right|\Delta_n \\ &\geqslant \frac{1}{2\sqrt{2}}\left(\frac{|y|-A_n}{|y|+\frac{1}{4}\sqrt{2}}\right)\Delta_n.\end{aligned}$$

Now, with $A_n > 0$, $$\frac{|y|-A_n}{|y|+\frac{1}{4}\sqrt{2}}$$

is a decreasing function of $|y|$; and, since

$$\frac{1}{2\sqrt{2}}\,\frac{4-A_n}{4+\frac{1}{4}\sqrt{2}} \to \frac{3+\frac{1}{2}\sqrt{2}}{1+8\sqrt{2}} > 1-\tfrac{1}{2}\sqrt{2},$$

it remains to consider the cases when $|y| = 0, 1, 2, 3$.

If $y = 0$, then $x = \pm 1$ since x, y are coprime and

$$|(L_1+c)L_2| = |\pm 1-2Q_n| \sim (1-\tfrac{1}{2}\sqrt{2})\Delta_n \quad \text{as } n \to \infty.$$

If $|y| = 1, 2, 3$, and $\min(|L_1+c|, |L_2|) < 1$, then we have, respectively,

$$\min(|L_1+c|, |L_2|) \geqslant \sqrt{2}-1,\ 3-2\sqrt{2},\ 3\sqrt{2}-4,$$

while

$$\max(|L_1+c|, |L_2|) \sim 2P_n \pm 2Q_n,\ 4P_n \pm 2Q_n,\ 6P_n \pm 2Q_n, \quad \text{as } n \to \infty.$$

But

$$(\sqrt{2}-1)(2P_n-2Q_n) \sim (1-\tfrac{1}{2}\sqrt{2})\Delta_n,$$

$$(3-2\sqrt{2})(4P_n-2Q_n) \sim (1-\tfrac{1}{2}\sqrt{2})\Delta_n,$$

$$(3\sqrt{2}-4)(6P_n-2Q_n) \sim (4\sqrt{2}-5)\Delta_n > (1-\tfrac{1}{2}\sqrt{2})\Delta_n, \quad \text{as } n \to \infty.$$

Hence, in all cases,

$$|(L_1+c)L_2| \geqslant c_n$$

for some constant $c_n \sim (1-\frac{1}{2}\sqrt{2})\Delta_n$ as $n \to \infty$. Thus, if n is sufficiently large, there is no solution of (3) in coprime integers x, y.

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CORRIGENDUM

TO 'NOTE ON A RESULT OF CHALK', BY H. DAVENPORT

[*Quart. J. of Math.* (Oxford) (2) 3 (1952) 130–8]

The result attributed to Chalk in the introductory paragraph of this paper has been found to be incorrect; for the correct results see J. H. H. Chalk immediately above. The substance of my paper is unaffected, except that Theorem 1 has less relevance to Chalk's problem than it appeared to have at the time.

ON L^2-SOLUTIONS OF THE WAVE EQUATION

By A. I. MARTIN (*Oxford*)

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1. It is known [(**1**); (**4**) § 10; (**8**) ch. 2, Theorem 5] that, if every solution of the ordinary differential equation

$$\frac{d^2\phi(x)}{dx^2}+\{\lambda-q(x)\}\phi(x) = 0, \tag{1.1}$$

where $q(x)$ is a real continuous function and x varies over the interval $0 \leqslant x < \infty$, is $L^2(0,\infty)$ for one value of the parameter λ, then every solution is $L^2(0,\infty)$ for any value of λ. In terms of the Green's function $G(x,\xi,\lambda)$ associated with the eigenvalue problem given by (1.1),

$$\phi(0)\cos\alpha+\phi'(0)\sin\alpha = 0, \tag{1.2}$$

and

$$0 < \int_0^\infty |\phi(x)|^2\,dx < \infty, \tag{1.3}$$

this result shows that, if $G(x,\xi,\lambda)$ is unique for one complex value of λ, then it is unique for every complex value of λ. Put in another way, we may say that, if the differential system given by (1.1), (1.2), and (1.3) has no solution for one complex† value of λ, then it has no solution for any complex value of λ. All this corresponds to the invariance of Weyl's limit-point case for complex values of λ. Similar results are known when x varies over the whole line $-\infty < x < \infty$.

The object of this paper is to show that corresponding results hold for the partial differential equation

$$\nabla^2\phi(x,y)+\{\lambda-q(x,y)\}\phi(x,y) = 0, \tag{1.4}$$

where $\nabla^2 = \partial^2/\partial x^2+\partial^2/\partial y^2$, $q(x,y)$ is a real continuous function, and the domain for x, y is the whole plane. The theory of eigenfunction expansions for (1.4) has been given by Titchmarsh (**7**). I shall use several facts out of this theory in the present analysis. Titchmarsh shows that there exists a Green's function $G(x,y,\xi,\eta,\lambda)$ corresponding to (1.4) and the whole (x,y)-plane; this function is an analytic function of λ, regular in the upper and lower halves of the complex λ-plane, it is symmetrical

† By complex, I mean that its imaginary part is not zero.

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with respect to (x, y) and (ξ, η), and satisfies the two inequalities†

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G(x,y,\xi,\eta,\lambda)|^2\, d\xi d\eta < \frac{K(x,y,u)}{|v|^2} \tag{1.5}$$

and

$$\int_{u_1}^{u_2} |\operatorname{im} G(x,y,\xi,\eta,\lambda)|\, du < K(x,y,\xi,\eta,u_1,u_2), \tag{1.6}$$

where $\lambda = u+iv$. The function

$$H(x,y,\xi,\eta,u) = \frac{1}{\pi}\lim_{v\to 0}\int_0^u \operatorname{im} G(x,y,\xi,\eta,u'+iv)\, du' \tag{1.7}$$

exists for all values of u, is of bounded variation in any finite interval, and satisfies

$$H(x,y,\xi,\eta,u) = \tfrac{1}{2}\{H(x,y,\xi,\eta,u+0)+H(x,y,\xi,\eta,u-0)\}, \tag{1.8}$$

and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{H(x,y,\xi,\eta,u)\}^2\, d\xi d\eta < K(x,y,u). \tag{1.9}$$

The Φ-function,

$$\Phi(x,y,\lambda,f) = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y,\xi,\eta,\lambda) f(\xi,\eta)\, d\xi d\eta, \tag{1.10}$$

where f belongs to L^2, is a solution of

$$\nabla^2\Phi + \{\lambda - q(x,y)\}\Phi = f, \tag{1.11}$$

and satisfies the inequality

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\Phi(x,y,\lambda)|^2\, dxdy \leqslant v^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x,y)|^2\, dxdy. \tag{1.12}$$

The relations (1.5)–(1.12) are all given in Titchmarsh's paper; they supply me with what I shall need here.

After having dealt with (1.4), I shall consider the corresponding non-homogeneous partial differential equation obtained from (1.4) by replacing the zero on the right-hand side by a function $f(x, y)$ of L^2; the results for (1.1), with the zero replaced by $f(x)$ of $L^2(0, \infty)$, have been stated by Weyl.

2. I shall say that $\psi = \psi(x, y)$ is an *L^2-solution* of (1.4) if it has continuous partial derivatives up to the second order, if $\phi = \psi$ satisfies (1.4), and if

$$0 < \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi(x,y)|^2\, dxdy < \infty. \tag{2.1}$$

† K denotes a function of the variables shown which is bounded when its variables lie in bounded regions.

For the complex parameter λ, let there be two distinct Green's functions; then their difference will yield an L^2-solution of (1.4). Conversely, if we have an L^2-solution $\psi(x,y)$, then $G(x,y,\xi,\eta,\lambda)+\psi(x,y)\psi(\xi,\eta)$, as well as G, would satisfy all the required properties of a Green's function. Hence, Green's function is unique if and only if there is no L^2-solution of (1.4).

THEOREM 1. *If* (1.4) *has no* L^2*-solution for one complex value of the parameter* λ*, then it has no* L^2*-solution for any complex value of* λ.

I shall give two proofs of this theorem. We cannot expect to extend this result, in a direct manner, so as to include all real values of λ since such a λ may be an eigenvalue (i.e. λ may lie in the point spectrum).

First Proof. Let (1.4) have no L^2-solution for $\lambda = \mu$ (complex), but suppose that we have $\phi = \psi(x,y)$ as an L^2-solution corresponding to $\lambda = \nu$ (complex). Let $\Phi(x,y,\mu)$ be the Φ-function (1.10) with $f = (\mu-\nu)\psi$. By using the differential equation for the Φ-function (1.11) it is readily seen that $\Phi(x,y,\mu)-\psi(x,y)$ is a solution of (1.4) when $\lambda = \mu$. But the squared modulus of this function is integrable over the whole plane and hence we must have this function identically equal to zero, i.e.

$$\Phi(x,y,\mu) = \psi(x,y).$$

Using (1.12), it is seen that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi|^2\,dxdy \leqslant |\mathrm{im}(\mu)|^{-2}|\mu-\nu|^2 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi|^2\,dxdy,$$

i.e., by (2.1), $|\nu-\mu| \geqslant |\mathrm{im}\,\mu|$. Thus ν lies outside a circle of centre μ and radius $|\mathrm{im}\,\mu|$. We conclude that, if (1.4) has no L^2-solution for $\lambda = \mu$, then it has no L^2-solution for any λ lying in this circle. If $\mathrm{im}\,\mu$ be taken positive (say), then, by proceeding from one circle to another (as in analytic continuation), we deduce that (1.4) has no L^2-solution for any λ lying in the upper half-plane. Since q is a real function, it is necessary only to take the conjugate value of equation (1.4) to deal with values of λ lying in the lower half-plane.

Second Proof. This depends on Green's formula for an infinite domain. Suppose that (1.4) has no L^2-solution for a particular complex value of λ. Let f, g be two arbitrary functions, which, together with $f^* = (q-\nabla^2)f$ and $g^* = (q-\nabla^2)g$, are L^2 over the whole plane. We have

$$(\nabla^2+\lambda-q)f = \lambda f - f^* = \theta, \quad \text{say.}$$

Subtracting this equation from (1.11), with $f = \theta$, we find that $f - \Phi(x, y, \lambda, \theta)$ satisfies (1.4) and must, therefore, be zero. Hence

$$f = \Phi(x, y, \lambda, \theta);$$

and similarly

$$g = \Phi(x, y, \lambda, \phi),$$

where $\phi = \lambda g - g^*$. Thus

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (fg^* - gf^*)\,dxdy &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \{\Phi(x, y, \lambda, \phi)\theta(x, y) - \Phi(x, y, \lambda, \theta)\phi(x, y)\}\,dxdy \\ &= 0. \end{aligned} \tag{2.2}$$

The last line follows by substituting the appropriate integral formula (1.10) for the Φ-function, and inverting an integration. To justify the inversion, let Δ_n be the region outside the square $-n \leqslant x, y \leqslant n$. Then it is sufficient to show that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \theta(x, y) \iint_{\Delta_n} G(x, y, \xi, \eta, \lambda)\phi(\xi, \eta)\,d\xi d\eta dxdy \to 0$$

as $n \to \infty$. From Schwarz's inequality and (1.12), the absolute value of this expression does not exceed

$$|v|^{-1}\left(\iint_{\Delta_n} |\phi|^2\,d\xi d\eta\right)^{\frac{1}{2}}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\theta|^2\,dxdy\right)^{\frac{1}{2}},$$

which tends to zero; (2.2) is Green's formula, which holds provided that (1.4) has no L^2-solution.

Suppose next that (1.4) has an L^2-solution $\psi = \psi_1 + i\psi_2$; then, by equating the real and imaginary parts of (1.4) separately to zero, we obtain

$$\psi_1^* = u\psi_1 - v\psi_2, \qquad \psi_2^* = u\psi_2 + v\psi_1 \quad (\lambda = u + iv).$$

Thus, $\psi_1\psi_2^* - \psi_2\psi_1^* = v|\psi|^2$; and the left-hand side of (2.2) does not vanish for the pair of functions $f = \psi_1$, $g = \psi_2$. We thus have the following theorem:

THEOREM 2. *Green's formula* (2.2) *holds for all functions f and g of the type considered if and only if the Green's function is unique.*

Since the left-hand side of (2.2) is independent of λ, Theorem 1 follows immediately from Theorem 2.

3. In the last section λ was restricted to take complex values only. Here I consider the case of real values; the result is suggested from the idea of 'eigenstates' which occur in quantum mechanics. I shall define the point spectrum to be those real values of λ for which the function

$$H(x,y,\xi,\eta,\lambda+0)-H(x,y,\xi,\eta,\lambda-0)$$

is not identically zero.

THEOREM 3. *Let the Green's function, corresponding to* (1.4) *and the whole* (x,y)*-plane, be unique for one complex value of* λ *(whence, by Theorem* 1, *it is also unique for any complex* λ*). Then* (1.4) *has an* L^2*-solution if and only if* λ *lies in the point spectrum.*

Suppose first that $\psi(x,y)$ is an L^2-solution for (1.4) when $\lambda=\mu$ (real). Clearly ψ may be taken to be a real function, for otherwise we need only consider its real, or imaginary, part. Let

$$\Phi(x,y,\lambda) = -(\lambda-\mu)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y,\xi,\eta,\lambda)\psi(\xi,\eta)\,d\xi d\eta,$$

where λ is complex. It is easily verified that $\Phi(x,y,\lambda)-\psi(x,y)$ is a solution of (1.4) and, hence, must be zero. Thus

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y,\xi,\eta,\lambda)\psi(\xi,\eta)\,d\xi d\eta = -\frac{\psi(x,y)}{\lambda-\mu}.$$

Taking imaginary parts in this equation, and writing $\lambda=u+iv$ $(v>0)$, we obtain, after integrating over $\mu-\epsilon\leqslant u\leqslant\mu+\epsilon$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left\{\int_{\mu-\epsilon}^{\mu+\epsilon}\operatorname{im} G(x,y,\xi,\eta,u+iv)\,du\right\}\psi(\xi,\eta)\,d\xi d\eta = \psi(x,y)\int_{-\epsilon/v}^{\epsilon/v}\frac{dt}{t^2+1},$$

uniform convergence justifying the inversion of the orders of integration on the left-hand side. With ϵ fixed and greater than zero make $v\to+0$; thus, by (1.7),

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\{H(x,y,\xi,\eta,\mu+\epsilon)-H(x,y,\xi,\eta,\mu-\epsilon)\}\psi(\xi,\eta)\,d\xi d\eta = \psi(x,y), \tag{3.1}$$

the justification for the inversion of the limiting process with the integration being similar to that used for (15.2) in (**7**). We may interpret (3.1) by saying that in the expansion formula of an L^2-solution of (1.4), with $\lambda=\mu$ (real), only those points of the spectrum lying in the immediate neighbourhood of μ make any contribution; for ϵ may be taken arbitrarily small. If μ is a point of continuity of the H-function, then,

by making $\epsilon \to 0$ in (3.1), we have $\psi(x, y) \equiv 0$; the inversion needed here is clearly allowable by the inequalities (1.6) and (1.9).

Next suppose that μ is a point of discontinuity of H (μ lies in the point spectrum); thus, by (1.8),

$$\phi(x, y, \xi, \eta) = H(x, y, \xi, \eta, \mu+0) - H(x, y, \xi, \eta, \mu-0) \tag{3.2}$$

is not identically zero and, using (1.9), we have

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \{\phi(x, y, \xi, \eta)\}^2 \, dxdy < \infty.$$

This function ϕ is symmetrical in (x, y) and (ξ, η), for this property, being true for Green's function, holds also for H. To complete the proof of Theorem 3, it is only necessary to show that, for fixed ξ and η, $\phi(x, y) = \phi(x, y, \xi, \eta)$ satisfies (1.4) with $\lambda = \mu$. With this aim in view, I shall next prove a lemma:

LEMMA 1. *Let the Green's function associated with* (1.4) *and the whole* (x, y)*-plane be unique; then*

$$(\lambda - \lambda') \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, y, s, t, \lambda) G(s, t, \xi, \eta, \lambda') \, dsdt$$
$$= G(x, y, \xi, \eta, \lambda) - G(x, y, \xi, \eta, \lambda'), \tag{3.3}$$

whenever $(x-\xi)^2 + (y-\eta)^2 \neq 0$, *and* λ, λ' *are any two complex numbers.*

Since the Green's function is L^2 at infinity, both sides of (3.3) exist; we have to prove equality. Using the differential equation (1.11), it is easy to verify that

$$(\lambda' - \lambda)\Phi(x, y, \lambda, \Phi(\lambda', f)) - \Phi(x, y, \lambda, f) + \Phi(x, y, \lambda', f)$$

satisfies (1.4). The last expression is L^2 over the whole plane [see (1.12)], and hence must be identically zero in the case of a unique Green's function; the first term is equal to

$$-(\lambda' - \lambda) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, y, s, t, \lambda) \Phi(s, t, \lambda', f) \, dsdt$$
$$= (\lambda' - \lambda) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(x, y, s, t, \lambda) G(s, t, \xi, \eta, \lambda') f(\xi, \eta) \, dsdtd\xi d\eta,$$

where we have used (1.10), uniform convergence justifying the inversion of the integrations in ξ, η and s, t. Thus, it is observed that we most certainly have equality when both sides of (3.3) are multiplied by $f(\xi, \eta)$ and integrated over the whole (ξ, η)-plane. Since f is any arbitrary function of L^2, it follows that both sides of (3.3) are indeed equal.

To show that $\phi(x,y)$ is a solution of (1.4), with $\lambda = \mu$, divide both sides of (3.3) by $\lambda - \lambda'$; and, from the resulting equation, subtract the corresponding equation with λ' replaced by $\bar{\lambda}'$. Since G takes conjugate values for conjugate values of its complex argument, we obtain, on writing $\lambda' = u' + iv'$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y,s,t,\lambda)\,\mathrm{im}\,G(s,t,\xi,\eta,\lambda')\,ds dt$$

$$= -\tfrac{1}{2}i\,G(x,y,\xi,\eta,\lambda)\left(\frac{1}{\lambda-\lambda'}-\frac{1}{\lambda-\bar{\lambda}'}\right)+$$

$$+\tfrac{1}{2}i\left\{\frac{G(x,y,\xi,\eta,\lambda')}{\lambda-\lambda'}-\frac{G(x,y,\xi,\eta,\bar{\lambda}')}{\lambda-\bar{\lambda}'}\right\}$$

$$= -\frac{\mathrm{im}\,G(x,y,\xi,\eta,\lambda')}{\lambda-\lambda'}+O\{|v'|\,|G(x,y,\xi,\eta,\lambda')|\}, \tag{3.4}$$

as $v' \to +0$, and λ is a fixed complex number. By an inequality [(7) 21],

$$\int_{\mu-\epsilon}^{\mu+\epsilon} |G(x,y,\xi,\eta,u'+iv')|\,du' = O(|v'|^{-\frac{1}{2}}).$$

Hence we obtain, on integrating (3.4) over $\mu-\epsilon \leqslant u' \leqslant \mu+\epsilon$,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y,s,t,\lambda)\left\{\int_{\mu-\epsilon}^{\mu+\epsilon} \mathrm{im}\,G(s,t,\xi,\eta,\lambda')\,du'\right\} ds dt$$

$$= \int_{u'=\mu-\epsilon}^{u'=\mu+\epsilon} \frac{1}{u'-\lambda}\,d\left[\int_0^{u'} \mathrm{im}\,G(x,y,\xi,\eta,u''+iv')\,du''\right]+O(|v'|^{\frac{1}{2}}).$$

By (1.6), the convergence of the right-hand side of (1.7) is bounded; and, hence, on making $v' \to +0$ in the last equation, we obtain†

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y,s,t,\lambda)\{H(s,t,\xi,\eta,\mu+\epsilon)-H(s,t,\xi,\eta,\mu-\epsilon)\}\,ds dt$$

$$= \int_{u'=\mu-\epsilon}^{u'=\mu+\epsilon} \frac{dH(x,y,\xi,\eta,u')}{u'-\lambda}. \tag{3.5}$$

Since ϕ is the saltus of H at the point μ, we may make $\epsilon \to 0$ to obtain

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(x,y,s,t,\lambda)\phi(s,t,\xi,\eta)\,ds dt = \frac{\phi(x,y,\xi,\eta)}{\mu-\lambda},$$

i.e.
$$\phi(x,y) = (\lambda-\mu)\Phi(x,y,\lambda,\phi). \tag{3.6}$$

† For the bounded convergence involved in Stieltjes' integral, see, for example, Widder, *The Laplace Transform*, 31, Theorem 16.4.

Hence, from (3.6),

$$\{\nabla^2+\mu-q(x,y)\}\phi(x,y) = (\lambda-\mu)\{\nabla^2+\lambda-q(x,y)\}\Phi(x,y,\lambda,\phi)- \\ -(\lambda-\mu)^2\Phi(x,y,\lambda,\phi) \\ = (\lambda-\mu)\phi-(\lambda-\mu)^2\Phi(x,y,\lambda,\phi) \\ = 0,$$

by (1.11), and again from (3.6). The proof of Theorem 3 is now complete.

Since the discontinuities of a function of bounded variation are enumerable, it follows that the real values of λ for which (1.4) has an L^2-solution are also enumerable. By using Green's formula (2.2), which holds for a unique Green's function, we observe that any two such solutions which correspond to two different real values of λ must be orthogonal. In the case when Green's function is not unique, it seems likely that (1.4) has an L^2-solution for every real value of λ; unfortunately, I have not so far been able to find any proof of this conjecture save in the simple case of $q(x,y) \equiv -C(x^2+y^2)^\alpha$ $(C>0, \alpha>1)$.

4. In this section I shall prove a theorem of comparison for two differential equations of the type (1.4).

THEOREM 4. *Consider the whole (x,y)-plane and the two equations*

$$\nabla^2\phi+\{\lambda-q_1(x,y)\}\phi = 0, \tag{4.1}$$

$$\nabla^2\phi+\{\lambda-q_2(x,y)\}\phi = 0, \tag{4.2}$$

where λ is complex. Let $q_1(x,y)-q_2(x,y)$ be bounded as $x^2+y^2\to\infty$; then, if one of the above equations has a unique Green's function, so has the other. Thus, if (4.1) *has no L^2-solution, then neither has* (4.2).

The one-dimensional case of this theorem is given in (**1**) and in (**9**) 266. In the case of the one-dimensional and one-sided infinite range $0 \leqslant x < \infty$ with no boundary condition at the left-hand end, the restricted view that λ must be complex may be relaxed to 'λ real or complex'.

Suppose it to be the case that (4.1) has no L^2-solution, whilst $\phi_2 = \phi_2(x,y)$ is an L^2-solution for (4.2). Since q_1-q_2 is bounded, and hence $(q_1-q_2)\phi_2$ is L^2 over the whole plane, we may form the Φ-function

$$\Phi_1(x,y,\lambda,(q_1-q_2)\phi_2) \\ = -\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_1(x,y,\xi,\eta,\lambda)\{q_1(\xi,\eta)-q_2(\xi,\eta)\}\phi_2(\xi,\eta)\,d\xi d\eta,$$

where G_1 is the Green's function for (4.1).

Apply the operator $\nabla^2+\lambda-q_1$ to the function

$$\Phi_1(x,y,\lambda,(q_1-q_2)\phi_2)+\phi_2(x,y).$$

The result is, by (1.11),

$$(q_1-q_2)\phi_2+(\nabla^2+\lambda-q_1)\phi_2 = (\nabla^2+\lambda-q_2)\phi_2 = 0.$$

Hence, since Φ_1 is L^2 and (4.1) has no L^2-solution, we have

$$\phi_2(x,y) = -\Phi_1(x,y,\lambda,(q_1-q_2)\phi_2);$$

and, by (1.12),

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\phi_2(x,y)|^2\,dxdy \leqslant A^2v^{-2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\phi_2(x,y)|^2\,dxdy,$$

where A is an upper bound for $|q_1-q_2|$. Hence, since $\psi = \phi_2$ satisfies (2.1), we have $v^2 \leqslant A^2$, and, by making $v \to \infty$, a contradiction follows. This proves Theorem 4.

In (**6**) it has been shown that, if

$$q(x,y) > -Ar^2-B \quad \text{as } r^2 = x^2+y^2 \to \infty,$$

where A, B are constants, then the Green's function for (1.4) and the whole plane is unique. Titchmarsh has also stated that it seems likely that this is a best-possible result for its kind. Sears (**3**) gives some generalizations of this uniqueness criterion. By using the result of Theorem 4, together with the corresponding result for the one-dimensional case, it can be shown that, if

$$q(x,y) = -Cr^\alpha+O(1) \quad \text{as } r^2 = x^2+y^2 \to \infty,$$

where $C > 0$ and $\alpha > 2$ are constants, then the Green's function for (1.4) and the whole plane is not unique.

5. I now consider the L^2-solutions of the equation

$$\nabla^2\phi+\{\lambda-q(x,y)\}\phi = f(x,y) \tag{5.1}$$

for real values of λ, where f is L^2 over the whole plane. It will be supposed that q gives rise to a unique Green's function; thus, in the case of complex values of λ, the L^2-solution of (5.1) is unique and given by (1.10).

Theorem 3 shows that the point spectrum can be defined in terms of the L^2-solutions of (1.4); (5.1) enables us to define the whole spectrum by means of L^2-solutions. I shall say that λ (real) is a *non-spectral point* if the H-function is constant in its neighbourhood; otherwise I shall say that λ *belongs to the spectrum*. The spectrum may be subdivided into the continuous and point spectra in accordance with the points of continuity and discontinuity of the H-function. The fundamental theorem is as follows:

THEOREM 5. *Let the Green's function be unique; then a point λ (real) is a non-spectral point if and only if* (5.1) *has a* (*unique*) L^2*-solution for every f of L^2.*

Weyl gave this result for the case of an ordinary differential equation and the interval $(0,\infty)$. In the next section I shall consider how the analytic continuation of the Green's function across the real λ-axis depends on the spectrum, and this yields the converse part of the theorem; for the direct result, it is necessary to use the idea of weak compactness for an infinite domain.

6. Analytic continuation of the Green's function across the real λ-axis

Let C be the square whose centre is at λ' (real) and whose sides are parallel to the λ-axes and of length $2l$. Divide C into three rectangles by means of the lines

$$\operatorname{im}\lambda = \pm\delta, \quad \lambda'-l \leqslant \operatorname{re}\lambda \leqslant \lambda'+l,$$

where $\delta > 0$; denote the uppermost rectangle by C_1, and the lowermost by C_2. Then, since $G(\lambda) = G(x,y,\xi,\eta,\lambda)$ is an analytic function of λ regular in the upper or lower half-planes, we have, for fixed λ inside C_1,

$$G(\lambda) = \frac{1}{2\pi i}\int_{C_1}\frac{G(z)}{z-\lambda}\,dz, \qquad 0 = \frac{1}{2\pi i}\int_{C_2}\frac{G(z)}{z-\lambda}\,dz.$$

Adding these two equations we obtain

$$G(\lambda) = \frac{1}{2\pi i}\int_{C_\delta}\frac{G(z)}{z-\lambda}\,dz + \frac{1}{2\pi i}\int_{\lambda'-l+i\delta}^{\lambda'+l+i\delta}\left\{\frac{G(z)}{z-\lambda} - \frac{\overline{G(\bar{z})}}{\bar{z}-\lambda}\right\}dz, \tag{6.1}$$

where C_δ is the part of C for which $|\operatorname{im}\lambda| \geqslant \delta$. Since Green's function assumes conjugate values for conjugate values of its complex argument, the second integral on the right-hand side of (6.1) is equal to

$$\int_{\lambda'-l+i\delta}^{\lambda'+l+i\delta}\left\{\frac{2i\operatorname{im}G(z)}{z-\lambda} + \left(\frac{1}{z-\lambda} - \frac{1}{\bar{z}-\lambda}\right)\overline{G(z)}\right\}dz = I_1 + I_2,$$

where

$$I_1 = 2i\int_{\lambda'-l+i\delta}^{\lambda'+l+i\delta}\frac{\operatorname{im}G(z)}{z-\lambda},$$

and

$$I_2 = -2i\delta\int_{\lambda'-l+i\delta}^{\lambda'+l+i\delta}\frac{\overline{G(z)}}{(z-\lambda)(\bar{z}-\lambda)}\,dz.$$

From (7) 21 we know that

$$\int_{u_1}^{u_2} |G(u+iv)|\,du = O(|v|^{-\frac{1}{2}})$$

as $v \to 0$; thus $I_2 = O(\delta^{\frac{1}{2}})$ as $\delta \to 0$. Also

$$I_1 = 2i \int_{\lambda'-l}^{\lambda'+l} \frac{\operatorname{im} G(u+i\delta)}{u+i\delta-\lambda}\,du = 2i \int_{\lambda'-l}^{\lambda'+l} \frac{\operatorname{im} G(u+i\delta)}{u-\lambda}\,du + O(\delta^{\frac{1}{2}})$$

$$= 2i \int_{\lambda'-l}^{\lambda'+l} \frac{1}{u-\lambda}\,d\left\{\int_0^u \operatorname{im} G(u'+i\delta)\,du'\right\} + O(\delta^{\frac{1}{2}})$$

$$\to 2\pi i \int_{\lambda'-l}^{\lambda'+l} \frac{1}{u-\lambda}\,dH(u)$$

as $\delta \to 0$, by a bounded-convergence property of Stieltjes' integral [see § 3]. Thus, on making $\delta \to 0$ in (6.1), we obtain:

LEMMA 2. *Let C be the square described above; then*

$$G(x,y,\xi,\eta,\lambda) = \frac{1}{2\pi i}\int_C \frac{G(x,y,\xi,\eta,z)}{z-\lambda}\,dz + \int_{u=\lambda'-l}^{u=\lambda'+l} \frac{dH(x,y,\xi,\eta,u)}{u-\lambda}, \quad (6.2)$$

where $0 < \operatorname{im}\lambda < l$, $\lambda'-l < \operatorname{re}\lambda < \lambda'+l$.

The formula (6.2) shows how the analytic continuation of G across the real λ-axis depends on the spectral form H and, hence, on the spectrum. For example, if λ' does not lie in the spectrum, then we have only to choose l so that $H(u)$ is constant in $\lambda'-l \leqslant u \leqslant \lambda'+l$ and it follows that G is regular (and single-valued) at and in the neighbourhood of λ', conjugate values being taken for conjugate values of λ. If λ' is an isolated point of the point spectrum and $\phi(x,y,\xi,\eta)$ the corresponding saltus of H, then the formula shows that G has a simple pole at λ' with residue $-\phi(x,y,\xi,\eta)$. We may make the following remark about the first integral on the right-hand side of (6.2). So far, we have only deduced its existence as a Cauchy principal value; however, for 'almost all' contours C, this integral is absolutely convergent and, hence, represents an analytic function of λ regular inside C. Thus, since

$$\int_0^1 \int_{u_1}^{u_2} |G(u+iv)|\,dudv$$

is finite [see (7) (2.4)], $\quad \int_0^1 |G(u+iv)|\,dv$

must be finite for almost all values of u, and, by taking C to be a square which cuts the real axis in points lying in this u-set, the required absolute convergence follows.

7. Let λ' be a point not lying in the spectrum; we wish to show that (5.1), with $\lambda = \lambda'$, has an L^2-solution which is provided by the analytic continuation of (1.10), for any f of L^2. Such a solution is obviously unique by Theorem 4. By (6.2),

$$G(x,y,\xi,\eta,\lambda) = \frac{1}{2\pi i}\int_C \frac{G(x,y,\xi,\eta,z)}{z-\lambda}\,dz,$$

for suitable contours C. Thus, by (1.10),

$$\Phi(x,y,\lambda) = \frac{1}{2\pi i}\int_C \frac{\Phi(x,y,z)}{z-\lambda}\,dz. \tag{7.1}$$

The inversion of the integration will be valid provided that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |G(x,y,\xi,\eta,u+iv)|^2\,d\xi d\eta = O(1) \tag{7.2}$$

as $v \to 0$ and u lies near λ'; furthermore, by (1.10), (7.2) shows that $\Phi(x,y,\lambda) = O(1)$ on C and hence, by (7.1), $\Phi(x,y,\lambda)$ is a regular function of λ in C. To prove (7.2), use Schwarz's inequality together with (1.5); thus

$$\left|\int_{-b}^{b}\int_{-b}^{b} \{G(x,y,\xi,\eta,u+iv) - G(x,y,\xi,\eta,i)\} f(\xi,\eta)\,d\xi d\eta\right| \leqslant \frac{K(x,y)}{|v|}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(\xi,\eta)|^2\,d\xi d\eta\right)^{\frac{1}{2}}.$$

The term contained in the modulus signs on the left-hand side of this inequality is a regular function of $u+iv$ on C. Hence, by Lemma 2.11 of (5), $K(x,y)/|v|$ may be replaced by $K_1(x,y)$ in the above inequality, where K_1 depends only on x and y and not on v. Taking f as the conjugate of $G(u+iv) - G(i)$ and making $b \to \infty$, we clearly get (7.2). Thus the inversion of the integration is justified, (7.1) holds, and $\Phi(x,y,\lambda)$ may be continued analytically across the real λ-axis in the neighbourhood of λ'. By a process similar to that applied to G it follows that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\Phi(x,y,\lambda)|^2\,dxdy$$

is finite and bounded for λ in C; thus, in particular, $\Phi(x,y,\lambda')$ is L^2.

From Lemma 1 it follows that, if λ, μ are any two complex numbers,

$$\Phi(x,y,\lambda,f) = \Phi(x,y,\mu,f) - (\lambda-\mu)\Phi(x,y,\mu,\Phi(\lambda));$$

and, by analytic continuation, this equation holds also for $\lambda = \lambda'$. Thus, by (1.11),

$$(\nabla^2+\mu-q)\Phi(x,y,\lambda',f) = f(x,y) - (\lambda'-\mu)\Phi(x,y,\lambda',f),$$

i.e.
$$(\nabla^2+\lambda'-q)\Phi(x,y,\lambda',f) = f(x,y). \tag{7.3}$$

With this, the proof of the statement made at the beginning of this section is complete.

8. By using the Green's formula of Theorem 2, we have the following result:

Let λ' lie in the point spectrum, and $\phi(x,y,\xi,\eta)$ be the corresponding saltus of H. Then (7.3), *where f is L^2, has no L^2-solution Φ unless f is orthogonal to $\phi(x,y,\xi,\eta)$ for every (ξ,η).*†

For

$$\begin{aligned}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\phi\,dxdy &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\lambda'\Phi\phi - \Phi^*\phi)\,dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (\lambda'\Phi\phi - \Phi\phi^*)\,dxdy \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Phi(\nabla^2\phi + (\lambda'-q)\phi)\,dxdy = 0.\end{aligned}$$

From this result, it is seen that, if (5.1), where λ is real, has an L^2-solution for every f of L^2, then such a solution is necessarily unique. For otherwise λ would lie in the point spectrum, and (5.1) would fail to have a solution of L^2 for f equal to the corresponding saltus.

9. Completion of the proof of Theorem 5

Suppose that λ is a real number such that for every f of L^2, (5.1), with $\lambda = \lambda'$, has an L^2-solution ϕ. Then, for given f, we know, from the preceding section, that ϕ is unique, and, hence, we may determine a least number $M = M(f)$, the bound of the transformation $f \to \phi$, such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\phi|^2\,dxdy \leqslant M(f)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f|^2\,dxdy. \tag{9.1}$$

† The corresponding result for a finite domain is well known. The proof of Theorem 3 shows that $\phi(x,y,\xi,\eta)$ generates the totality of L^2-solutions of $\nabla^2\Phi + (\lambda'-q)\Phi = 0$.

We wish to show that, for all functions f of L^2, $M(f)$ is bounded. Suppose, on the contrary, that $\limsup_{f \in L^2} M(f)$ is $+\infty$; then we may choose a sequence of functions f_n such that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f_n|^2\,dxdy = 1 \text{ (for each } n), \quad \text{and} \quad \lim_{n\to\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\phi_n|^2\,dxdy = +\infty.$$

The sequence f_n is weakly compact,† i.e. there exists an f of L^2 and a sequence of integers $n_1, n_2, \ldots$ such that

$$\lim_{\nu\to\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{n_\nu} g\,dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} fg\,dxdy,$$

for any g of L^2. Let ψ be the L^2-solution of our partial differential equation which corresponds to g; then, by the Green's formula of Theorem 2, we have

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{n_\nu} g\,dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{n_\nu}\psi\,dxdy;$$

and hence

$$\lim_{\nu\to\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi_{n_\nu} g\,dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\psi\,dxdy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \phi g\,dxdy,$$

again by using Green's formula, and writing ϕ for the L^2-solution corresponding to f. Hence, ϕ_{n_ν} is weakly convergent to ϕ, and, by Corollary 2 of Theorem I on page 46 of Murray's book, it follows that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\phi_{n_\nu}|^2\,dxdy$$

is bounded. This is a contradiction; hence, $\limsup_{f \in L^2} M(f)$ must be finite. We thus see that (9.1) may be replaced by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\phi|^2\,dxdy \leqslant M(\lambda') \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f|^2\,dxdy, \tag{9.2}$$

where $M(\lambda')$ is independent of f but may depend on λ'. We take $M(\lambda')$ as the least number with this property.

For complex λ, let $M(\lambda)$ be the bound of the transformation, $f \to \Phi$, given by (1.10); thus, by (1.12), $M(\lambda) \leqslant 1/v^2$.

† See Murray, *Linear Transformations in Hilbert Space* (Princeton, 1941), 47, Theorem III.

From equation (1.11),

$$\nabla^2\Phi+\{\lambda'-q(x,y)\}\Phi = f+(\lambda'-\lambda)\Phi,$$

and we may replace f by $f+(\lambda'-\lambda)\Phi$ and ϕ by Φ in (9.2). Thus,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Phi|^2\,dxdy \leqslant M(\lambda')\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f+(\lambda'-\lambda)\Phi|^2\,dxdy$$

$$\leqslant 2M(\lambda')\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f|^2\,dxdy+|\lambda'-\lambda|^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Phi|^2\,dxdy\right\};$$

and we obtain, for $|\lambda'-\lambda|$ small,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\Phi|^2\,dxdy \leqslant \frac{2M(\lambda')}{1-2|\lambda'-\lambda|^2M(\lambda')}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f|^2\,dxdy.$$

Thus, $M(\lambda)$ does not exceed $2M(\lambda')/\{1-2|\lambda'-\lambda|^2M(\lambda')\}$, and, hence, is bounded for all complex values of λ in the neighbourhood of λ'. Taking $f(x,y) = G(x,y,\xi,\eta,i)$, (3.3), with $\lambda' = i$, we see that

$$\Phi(x,y,\lambda,f) = \frac{G(x,y,\xi,\eta,\lambda)-G(x,y,\xi,\eta,i)}{\lambda-i};$$

and thus

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|G(x,y,\xi,\eta,\lambda)-G(x,y,\xi,\eta,i)|^2\,dxdy$$

$$\leqslant M(\lambda)|\lambda-i|^2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|G(x,y,\xi,\eta,i)|^2\,dxdy.$$

Hence $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|G(x,y,\xi,\eta,\lambda)|^2\,dxdy$ is bounded for λ near λ'; and again using (3.3), this time with $\lambda' = \bar{\lambda}$, we have

$$\operatorname{im} G(x,y,\xi,\eta,\lambda) = O(\operatorname{im}\lambda), \tag{9.3}$$

as $\operatorname{im}\lambda \to 0$, uniformly for $\operatorname{re}\lambda$ in a neighbourhood of λ'. From (9.3) it follows that the H-function is constant in an interval containing λ'; and the proof of Theorem 5 is complete.

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ON THE ASYMPTOTIC DISTRIBUTION OF EIGENVALUES

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1. LET
$$\frac{d^2\psi}{dx^2}+\{\lambda-q(x)\}\psi = 0 \tag{1.1}$$
be a given differential equation, in which $q(x)$ is continuous and tends steadily to infinity as $x \to \infty$ and as $x \to -\infty$. Such an equation has a discrete spectrum, with eigenvalues λ_0, λ_1,... and eigenfunctions
$$\psi_0(x), \psi_1(x),\ldots .$$

Let X and X' be the positive and negative roots of the equation $q(x) = \lambda$, where λ is real and greater than the minimum of $q(x)$. Thus X and X' are functions of λ. The large eigenvalues are then roots of an equation of the form
$$\int_{X'}^{X} \{\lambda-q(x)\}^{\frac{1}{2}}\,dx = (n+\tfrac{1}{2})\pi+o(1), \tag{1.2}$$
where n is an integer.

This formula has been much discussed in books on quantum mechanics: see for example Kemble (**1**) Ch. III, the actual formula being (21.10). I reconsider it here because none of the existing methods seems quite satisfactory. The method of Zwaan, given by Kemble, assumes that $q(x)$ is an analytic function of x, and that certain functions related to (1.1) have favourable properties on certain paths in the complex plane. The method of Langer [see Kemble (**1**) 95, the references to Langer in Titchmarsh (**3**), and Langer (**2**)] can be put into a form in which it is not necessary to assume that $q(x)$ is analytic; but, as I shall show later, it leaves in doubt the relation between the integer n which appears in (1.2) and the integer which is the suffix of the eigenvalue that satisfies (1.2). In Ch. VIII of my book (**3**) I gave another method which solves this problem, but only under the assumption that $q(x)$ is an integral function of x; the details are worked out only in the case $q(x) = x^4$.

In the present paper I assume that $q(x)$ has derivatives up to the third order satisfying certain conditions of inequality. I show that (1.2) holds

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with $\lambda = \lambda_n$ and an error-term of a certain order. I consider also the corresponding problem involving the interval $0 \leqslant x < \infty$, with a given boundary condition at $x = 0$.

2. I consider first the case where the interval is $(0, \infty)$, and begin by obtaining some approximate formulae for solutions of (1.1). Let

$$\xi(x) = \int_0^x \{\lambda - q(t)\}^{\frac{1}{2}}\, dt, \qquad \eta(x) = \{\lambda - q(x)\}^{\frac{1}{4}}\psi(x),$$

where, if $0 < \arg\lambda \leqslant \pi$,

$$0 < \arg\{\lambda - q(t)\}^p \leqslant p\pi \quad (p = \tfrac{1}{4}, \tfrac{1}{2}).$$

The equation (1.1) is then equivalent to the integral equation

$$\eta(x) = e^{i\xi(x)} + \int_x^\infty R(t)\eta(t)\sin\{\xi(t) - \xi(x)\}\, dt, \tag{2.1}$$

where

$$R(t) = -\frac{q''(t)}{4\{\lambda - q(t)\}^{\frac{3}{2}}} - \frac{5q'^2(t)}{16\{\lambda - q(t)\}^{\frac{5}{2}}}. \tag{2.2}$$

We shall assume that $q(x)$ and $q'(x)$ are non-decreasing, and that, as $x \to \infty$,

$$q'(x)/q(x) = O(x^{-1}). \tag{2.3}$$

On integrating this, it is seen to involve

$$q(ax) = O\{q(x)\} \tag{2.4}$$

for any fixed a greater than 1.

Let $\lambda = \rho e^{i\gamma}$, where $0 < \gamma_0 \leqslant \gamma \leqslant \pi$. Then

$$|\lambda - q(t)|^2 = \rho^2 + q^2(t) - 2\rho q(t)\cos\gamma \geqslant (1 - \cos\gamma_0)\{\rho^2 + q^2(t)\},$$

and so

$$|\lambda - q(t)| > A\{\rho + q(t)\}.$$

Let X be defined by $q(X) = \rho$. Then

$$\int_0^\infty \frac{q'^2(t)\, dt}{|\lambda - q(t)|^{\frac{5}{2}}} < Aq'(X)\int_0^X \frac{q'(t)\, dt}{\rho^{\frac{5}{2}}} + \frac{A}{X}\int_X^\infty \frac{q'(t)\, dt}{\{q(t)\}^{\frac{3}{2}}}$$

$$= O\{q'(X)\rho^{-\frac{3}{2}}\} + O(X^{-1}\rho^{-\frac{1}{2}}) = O(X^{-1}\rho^{-\frac{1}{2}}).$$

Also

$$\int_0^\infty \left|\frac{q''(t)}{\{\lambda - q(t)\}^{\frac{3}{2}}}\right| dt < A\int_0^\infty \frac{q''(t)\, dt}{\{\rho + q(t)\}^{\frac{3}{2}}}$$

$$= A\left\{\left[\frac{q'(t)}{\{\rho + q(t)\}^{\frac{3}{2}}}\right]_0^\infty + \frac{3}{2}\int_0^\infty \frac{q'^2(t)\, dt}{\{\rho + q(t)\}^{\frac{5}{2}}}\right\}$$

$$= O(\rho^{-\frac{3}{2}}) + O(X^{-1}\rho^{-\frac{1}{2}}) = O(X^{-1}\rho^{-\frac{1}{2}})$$

since $q(x) > Ax$. *Hence, as* $|\lambda| \to \infty$,

$$\int_0^\infty |R(t)|\,dt = O(X^{-1}\rho^{-\frac{1}{2}}). \tag{2.5}$$

The equation (2.1) can now be solved by iteration: details are given, for example, in my paper (**4**). We obtain a solution $\eta(x)$ such that

$$\eta(x) = e^{i\xi(x)} + \int_x^\infty R(t)\sin\{\xi(x)-\xi(t)\}e^{i\xi(t)}\,dt + O(|e^{i\xi(x)}|X^{-2}\rho^{-1}) \tag{2.6}$$

as $\rho \to \infty$, uniformly for $x \geqslant 0$.

Suppose in particular that λ is real and negative, say $\lambda = -\mu$. Then

$$\xi(x) = i\int_0^x \{\mu+q(u)\}^{\frac{1}{2}}\,du.$$

Since

$$R(t) = O\left\{\frac{q(t)}{\{\mu+q(t)\}^{\frac{3}{2}}}\right\} + O\left\{\frac{q^2(t)}{\{\mu+q(t)\}^{\frac{5}{2}}}\right\} = O\left\{\frac{1}{\{\mu+q(t)\}^{\frac{1}{2}}}\right\},$$

we have

$$\begin{aligned}\int_x^\infty e^{2i\xi(t)}R(t)\,dt &= O\left[\int_0^\infty \{\mu+q(t)\}^{-\frac{1}{2}}\exp\left\{-2\int_0^t \{\mu+q(u)\}^{\frac{1}{2}}\,du\right\}dt\right]\\ &= O\left[\mu^{-1}\int_x^\infty \{\mu+q(t)\}^{\frac{1}{2}}\exp\left\{-2\int_0^t \{\mu+q(u)\}^{\frac{1}{2}}\,du\right\}dt\right]\\ &= O\left[\mu^{-1}\exp\left\{-2\int_0^x \{\mu+q(u)\}^{\frac{1}{2}}\,du\right\}\right].\end{aligned}$$

Hence (2.6) gives

$$\eta(x) = e^{i\xi(x)}\left\{1+\tfrac{1}{2}i\int_x^\infty R(t)\,dt + O(\mu^{-1})\right\}. \tag{2.7}$$

The corresponding solution of (1.1) therefore satisfies

$$\psi(x,-\mu) = \{\mu+q(x)\}^{-\frac{1}{4}}e^{i\xi(x)-\frac{1}{4}i\pi}\left\{1+\tfrac{1}{2}i\int_x^\infty R(t)\,dt + O(\mu^{-1})\right\}. \tag{2.8}$$

3. Consider next the solution $\phi(x,\lambda)$ of (1.1) which satisfies the boundary conditions

$$\phi(0,\lambda) = \sin\alpha, \qquad \phi'(0,\lambda) = -\cos\alpha.$$

Let $\eta_1(x,\lambda) = \{\lambda - q(x)\}^{\frac{1}{4}}\phi(x,\lambda)$. Then η_1 satisfies the integral equation

$$\eta_1(x) = \eta_1(0)\cos\xi(x) + \eta_1'(0)\{\lambda - q(0)\}^{-\frac{1}{2}}\sin\xi(x) + \\ + \int_0^x R(t)\eta_1(t)\sin\{\xi(x) - \xi(t)\}\,dt. \qquad (3.1)$$

This also can be solved by iteration. Suppose for example that $\alpha = 0$. Then the solution $\eta_1(x)$ satisfies

$$\eta_1(x) = -\{\lambda - q(0)\}^{-\frac{1}{4}}\Big\{\sin\xi(x) + \int_0^x \sin\{\xi(x) - \xi(t)\}\sin\xi(t)R(t)\,dt + \\ + O(|e^{-i\xi(x)}|X^{-2}\rho^{-1})\Big\}. \qquad (3.2)$$

As before, if $\lambda = -\mu$, this can be reduced to

$$\eta_1(x) = -\{-\mu - q(0)\}^{-\frac{1}{4}}\Big\{\sin\xi(x) - \tfrac{1}{2}\cos\xi(x)\int_0^x R(t)\,dt + O|e^{-i\xi(x)}|\mu^{-1}\Big\}. \qquad (3.3)$$

Hence

$$\phi(x,-\mu) = i\{\mu + q(0)\}^{-\frac{1}{4}}\{\mu + q(x)\}^{-\frac{1}{4}}\Big\{\sin\xi(x) - \tfrac{1}{2}\cos\xi(x)\int_0^x R(t)\,dt + \\ + O(|e^{-i\xi(x)}|\mu^{-1})\Big\}. \qquad (3.4)$$

4. If λ is real and positive, the above analysis fails since $R(t)$ is infinite at $t = X$. The method of Langer therefore has to be used. This goes in outline as follows. Let

$$\zeta = \zeta(x) = \int_X^x \{\lambda - q(t)\}^{\frac{1}{2}}\,dt,$$

where

$$\arg\zeta = \begin{cases} \frac{1}{2}\pi & (x > X), \\ -\pi & (x < X), \end{cases}$$

X being now defined by $q(X) = \lambda$. Defining $\eta(x)$ as before, we can transform (1.1) into

$$\frac{d^2\eta}{d\zeta^2} + \Big(1 + \frac{5}{36\zeta^2}\Big)\eta = f(x)\eta, \qquad (4.1)$$

where

$$f(x) = \frac{5}{36\zeta^2} - \frac{q''(x)}{4\{\lambda - q(x)\}^2} - \frac{5q'^2(x)}{16\{\lambda - q(x)\}^3}.$$

Now (4.1) is formaliy equivalent to the integral equation

$$\eta(x) = (\tfrac{1}{2}\pi\zeta)^{\frac{1}{2}}H^{(1)}_{\frac{1}{3}}(\zeta)+\tfrac{1}{2}\pi i\int_x^\infty \{H^{(1)}_{\frac{1}{3}}(\zeta)J_{\frac{1}{3}}(\theta)-J_{\frac{1}{3}}(\zeta)H^{(1)}_{\frac{1}{3}}(\theta)\}\times$$
$$\times\zeta^{\frac{1}{2}}\theta^{\frac{1}{2}}f(t)\eta(t)\{\lambda-q(t)\}^{\frac{1}{2}}\,dt, \qquad (4.2)$$

where $\theta = \zeta(t)$.

In order to solve this, we make the additional assumptions that, as $x\to\infty$,

$$\frac{q''(x)}{q'(x)} = O\left(\frac{1}{x}\right), \qquad \frac{q'''(x)}{q''(x)} = O\left(\frac{1}{x}\right). \qquad (4.3)$$

As in § 2, these involve

$$q'(ax) = O\{q'(x)\}, \qquad q''(ax) = O\{q''(x)\}, \qquad (4.4)$$

for any fixed a greater than 1.

We shall now prove that, as $\lambda\to\infty$,

$$\int_0^\infty |f(x)|\,|\lambda-q(x)|^{\frac{1}{2}}\,dx = O(X^{-1}\lambda^{-\frac{1}{2}}). \qquad (4.5)$$

As far as the ranges $x < \frac{1}{2}X$ and $x > 2X$ are concerned, this follows from the analysis of § 2, together with the results

$$q(X) = \int_0^X q'(t)\,dt + q(0) < AXq'(X),$$

$$|\zeta(2X)| \geqslant \int_X^{2X}\{(x-X)q'(X)\}^{\frac{1}{2}}\,dx = \tfrac{2}{3}X^{\frac{3}{2}}\{q'(X)\}^{\frac{1}{2}} > AX\{q(X)\}^{\frac{1}{2}},$$

$$\int_{2X}^\infty \frac{\{\lambda-q(x)\}^{\frac{1}{2}}}{\zeta^2}\,dx = \frac{1}{\zeta(2X)} = O(X^{-1}\lambda^{-\frac{1}{2}}),$$

with a similar argument for the interval $(0, \frac{1}{2}X)$.

Next let $X\leqslant x\leqslant 2X$. Writing

$$-i\,\zeta(x) = \int_X^x \frac{q'(t)\{q(t)-\lambda\}^{\frac{1}{2}}}{q'(t)}\,dt$$

and integrating by parts twice, we obtain

$$-i\,\zeta(x) = \frac{2\{q(x)-\lambda\}^{\frac{3}{2}}}{3q'(x)}\left\{1+\frac{2\{q(x)-\lambda\}q''(x)}{5q'^2(x)}-S\right\},$$

where

$$S = \frac{2q'(x)}{\{q(x)-\lambda\}^{\frac{3}{2}}}\int_X^x \{q(t)-\lambda\}^{\frac{5}{2}}\frac{d}{dt}\left\{\frac{q''(t)}{q'^3(t)}\right\}dt.$$

Hence

$$-\frac{1}{\zeta^2(x)} = \frac{9q'^2(x)}{4\{q(x)-\lambda\}^3}\left[1-\frac{4\{q(x)-\lambda\}q''(x)}{5q'^2(x)}+O\left\{\frac{\{q(x)-\lambda\}^2q''^2(x)}{q'^4(x)}\right\}+\right.$$
$$\left.+O(|S|)\right],$$

and so

$$\int_X^{2X}|f(x)||\lambda-q(x)|^{\frac{1}{2}}\,dx = O\left\{\int_X^{2X}\frac{q''^2(x)\,dx}{\{q(x)-\lambda\}^{\frac{1}{2}}q'^2(x)}\right\}+$$
$$+O\left\{\int_X^{2X}\frac{q'^3(x)\,dx}{\{q(x)-\lambda\}^4}\int_X^x\{q(t)-\lambda\}^{\frac{5}{2}}\left|\frac{d}{dt}\left\{\frac{q''(t)}{q'^3(t)}\right\}\right|dt\right\}.$$

The first term on the right is

$$O\left\{\frac{1}{X^2q'(X)}\int_X^{2X}\frac{q'(x)\,dx}{\{q(x)-\lambda\}^{\frac{1}{2}}}\right\} = O\left\{\frac{\{q(X)\}^{\frac{1}{2}}}{X^2q'(X)}\right\} = O(X^{-1}\lambda^{-\frac{1}{2}}),$$

and the second term is

$$O\left\{\frac{1}{X^2q'(X)}\int_X^{2X}\frac{q'(x)\,dx}{\{q(x)-\lambda\}^4}\int_X^x\{q(t)-\lambda\}^{\frac{5}{2}}q'(t)\,dt\right\}$$
$$= O\left\{\frac{1}{X^2q'(X)}\int_X^{2X}\frac{q'(x)\,dx}{\{q(x)-\lambda\}^{\frac{1}{2}}}\right\} = O(X^{-1}\lambda^{-\frac{1}{2}})$$

similarly. Similar arguments also apply to the interval $(\frac{1}{2}X, X)$, and so (4.5) follows.

We can now solve (4.2) by iteration [cf. (**4**) § 3]. The solution satisfies

$$\eta(x) = (\tfrac{1}{2}\pi\zeta)^{\frac{1}{2}}H^{(1)}_{\frac{1}{3}}(\zeta)+O(X^{-1}\lambda^{-\frac{1}{2}}e^{-\mathrm{im}\zeta}) \tag{4.6}$$

as $\lambda\to\infty$, uniformly for $x\geqslant 0$.

In particular, let $x=0$. Then $\zeta = ze^{-i\pi}$, where

$$z = z(\lambda) = \int_0^X\{\lambda-q(t)\}^{\frac{1}{2}}\,dt$$

is real and positive; and

$$H^{(1)}_{\frac{1}{3}}(\zeta) = (-2i/\sqrt{3})e^{\frac{1}{3}i\pi}\{J_{\frac{1}{3}}(z)+J_{-\frac{1}{3}}(z)\}$$
$$= 2e^{-\frac{1}{6}i\pi}(2/\pi)^{\frac{1}{2}}z^{-\frac{1}{2}}\{\cos(z-\tfrac{1}{4}\pi)+O(z^{-1})\}.$$

Hence $$\psi(0,\lambda) = 2e^{-\frac{2}{3}i\pi}\{\lambda-q(0)\}^{-\frac{1}{4}}\{\cos(z-\tfrac{1}{4}\pi)+O(z^{-1})\}. \tag{4.7}$$

Similarly, by differentiating (4.2) with respect to x, using (4.6), and then putting $x = 0$, we obtain

$$\psi'(0,\lambda) = 2e^{-\frac{2}{3}i\pi}\{\lambda-q(0)\}^{\frac{1}{4}}\{\sin(z-\tfrac{1}{4}\pi)+O(z^{-1})\}. \tag{4.8}$$

5. In the case of the problem with $0 \leqslant x < \infty$ and boundary condition $\psi(0) = 0$, the eigenvalues λ_n are the roots of the equation $\psi(0,\lambda) = 0$, and so they satisfy the relation

$$\cos\{z(\lambda_n)-\tfrac{1}{4}\pi\} = O\{1/z(\lambda_n)\}. \tag{5.1}$$

Hence $\pi^{-1}z(\lambda_n)-\frac{3}{4}$ is approximately equal to an integer, m say, and in fact

$$z(\lambda_n) = (m+\tfrac{3}{4})\pi+O(1/m). \tag{5.2}$$

This is just Langer's result, with a particular form of the error term. However, there is nothing in this argument to determine the relation between the integers m and n. All that can be said definitely so far is the following: *let λ'_m be the value of λ for which $z(\lambda) = (m+\frac{1}{4})\pi$. Then, by* (4.7), *each interval $(\lambda'_m, \lambda'_{m+1})$, starting from some value $m = m_0$, contains at least one eigenvalue, and any such eigenvalue satisfies* (5.2).

6. We shall now consider the Green's function $G(x,y,\lambda)$ associated with the above problem, particularly in the case where $y = x$ and λ is real and negative, say $\lambda = -\mu$. It will be assumed in the first place that

$$\int^{\infty} \{q(x)\}^{-\frac{1}{2}}\,dx \tag{6.1}$$

is convergent.

Let $\phi(x,\lambda)$ and $\psi(x,\lambda)$ be the functions so denoted in § 3, with $\alpha = 0$. Then the Wronskian of ϕ and ψ is equal to $\psi(0,\lambda)$, and so, if $x \leqslant y$,

$$G(x,y,\lambda) = -\phi(x,\lambda)\psi(y,\lambda)/\psi(0,\lambda).$$

It follows from (2.8) and (3.4) that

$$G(x,x,-\mu) = -i\{\mu+q(x)\}^{-\frac{1}{2}}\left\{e^{i\xi(x)}\sin\xi(x)-\tfrac{1}{2}e^{2i\xi(x)}\int_0^x R(t)\,dt+O(\mu^{-1})\right\}$$

as $\mu \to \infty$, uniformly with respect to x. Hence

$$\int_0^{\infty} G(x,x,-\mu)\,dx = -i\int_0^{\infty}\frac{e^{i\xi(x)}\sin\xi(x)}{\{\mu+q(x)\}^{\frac{1}{2}}}\,dx+O\left\{\mu^{-\frac{1}{2}}\int_0^{\infty}\frac{e^{2i\xi(x)}}{\{\mu+q(x)\}^{\frac{1}{2}}}\,dx\right\}+$$

$$+O\left\{\mu^{-1}\int_0^{\infty}\frac{dx}{\{\mu+q(x)\}^{\frac{1}{2}}}\right\}. \tag{6.2}$$

The last term is clearly $o(\mu^{-1})$. Also

$$\int_0^\infty \frac{e^{2i\xi(x)}}{\{\mu+q(x)\}^{\frac{1}{2}}}\,dx = \left[\frac{-\frac{1}{2}\exp\left\{-2\int_0^x \{\mu+q(t)\}^{\frac{1}{2}}\,dt\right\}}{\mu+q(x)}\right]_0^\infty +$$

$$+\frac{1}{2}\int_0^\infty \frac{q'(x)}{\{\mu+q(x)\}^2}\exp\left\{-2\int_0^x \{\mu+q(t)\}^{\frac{1}{2}}\,dt\right\}dx$$

$$= \frac{1}{2\{\mu+q(0)\}} + O\left[\mu^{-\frac{3}{2}}\int_0^\infty \{\mu+q(x)\}^{\frac{1}{2}}\exp\left\{-2\int_0^x \{\mu+q(t)\}^{\frac{1}{2}}\,dt\right\}dx\right]$$

$$= \tfrac{1}{2}\mu^{-1}+O(\mu^{-\frac{3}{2}}).$$

Hence
$$\int_0^\infty G(x,x,-\mu)\,dx = \frac{1}{2}\int_0^\infty \frac{dx}{\{\mu+q(x)\}^{\frac{1}{2}}} - \frac{1}{4\mu} + o\left(\frac{1}{\mu}\right). \tag{6.3}$$

Now let λ and λ' be any two numbers not in the spectrum. Then

$$G(x,y,\lambda)-G(x,y,\lambda') = (\lambda-\lambda')\int_0^\infty G(x,t,\lambda)G(t,y,\lambda)\,dt$$

$$= (\lambda-\lambda')\sum_{n=0}^\infty \frac{\psi_n(x)\psi_n(y)}{(\lambda_n-\lambda)(\lambda_n-\lambda')}. \tag{6.4}$$

This follows from (2.13.1) of my book, and the Parseval formula. Let $\lambda = -\mu$, $\lambda' = -\mu'$, where μ and μ' are real and positive. On taking $y = x$, integrating with respect to x, and using the fact that $G(x,x,-\mu)$ is $L(0,\infty)$, we see that $\sum \lambda_n^{-2}$ is convergent, and then that

$$\int_0^\infty G(x,x,-\mu)\,dx - \int_0^\infty G(x,x,-\mu')\,dx = (\mu'-\mu)\sum_{n=0}^\infty \frac{1}{(\lambda_n+\mu)(\lambda_n+\mu')}. \tag{6.5}$$

Now let $\mu' \to \infty$. The term on the left involving μ' tends to zero. Hence the right-hand side is bounded,

$$\sum_{\lambda_n<\mu'} \frac{1}{\lambda_n+\mu}$$

is bounded, $\sum \lambda_n^{-1}$ is convergent, and so finally

$$\int_0^\infty G(x,x,-\mu)\,dx = \sum_{n=0}^\infty \frac{1}{\lambda_n+\mu}. \tag{6.6}$$

Let $N(t)$ be the number of eigenvalues not exceeding t. Then, by (6.3) and (6.6),

$$\int_{\lambda_0}^{\infty} \frac{N(t)\,dt}{(t+\mu)^2} = \sum_{n=0}^{\infty} \int_{\lambda_n}^{\lambda_{n+1}} \frac{(n+1)\,dt}{(t+\mu)^2} = \sum_{n=0}^{\infty} \frac{1}{\lambda_n+\mu}$$

$$= \frac{1}{2}\int_0^{\infty} \frac{dx}{\{\mu+q(x)\}^{\frac{1}{2}}} - \frac{1}{4\mu} + o\left(\frac{1}{\mu}\right). \quad (6.7)$$

7. Let the number of eigenvalues in the interval $(\lambda'_m, \lambda'_{m+1})$, where $m \geqslant m_0$, be p_m, so that p_m is a positive integer. Then, if $\lambda'_m \leqslant t < \lambda'_{m+1}$, $m \geqslant m_0$,

$$\begin{aligned} N(t) \geqslant N(\lambda'_m) &\geqslant p_{m_0} + \ldots + p_{m-1} \\ &= \pi^{-1} z(\lambda'_{m+1}) - m - \tfrac{5}{4} + p_{m_0} + \ldots + p_{m-1} \\ &\geqslant \pi^{-1} z(t) + P(t) - m_0 - \tfrac{5}{4}, \end{aligned}$$

where

$$P(t) = \sum_{\nu=m_0}^{m-1} (p_\nu - 1)$$

is a non-negative non-decreasing function of t. Hence

$$\int_a^{\infty} \frac{N(t)\,dt}{(t+\mu)^2} \geqslant \frac{1}{\pi}\int_a^{\infty} \frac{z(t)\,dt}{(t+\mu)^2} + \int_a^{\infty} \frac{P(t)\,dt}{(t+\mu)^2} - \frac{m_0+\frac{5}{4}}{a+\mu},$$

where $a = \lambda'_{m_0}$. We can replace a by 0 with error $O(\mu^{-2})$. Now

$$\int_0^{\infty} \frac{z(t)\,dt}{(t+\mu)^2} = \int_0^{\infty} \frac{dt}{(t+\mu)^2} \int_{q<t} \{t-q(x)\}^{\frac{1}{2}}\,dx$$

$$= \int_0^{\infty} dx \int_q^{\infty} \frac{(t-q)^{\frac{1}{2}}}{(t+\mu)^2}\,dt.$$

The inner integral is equal to

$$\int_0^{\infty} \frac{u^{\frac{1}{2}}\,du}{(u+\mu+q)^2} = \frac{1}{(\mu+q)^{\frac{1}{2}}} \int_0^{\infty} \frac{v^{\frac{1}{2}}\,dv}{(v+1)^2} = \frac{\pi}{2(\mu+q)^{\frac{1}{2}}}.$$

Hence

$$\int_0^{\infty} \frac{N(t)\,dt}{(t+\mu)^2} \geqslant \frac{1}{2}\int_0^{\infty} \frac{dx}{\{\mu+q(x)\}^{\frac{1}{2}}} + \int_a^{\infty} \frac{P(t)\,dt}{(t+\mu)^2} - \frac{A}{\mu}.$$

From this and (6.6) it follows that

$$\frac{A}{\mu} > \int_a^\infty \frac{P(t)\,dt}{(t+\mu)^2} \geqslant \int_\mu^\infty \frac{P(t)\,dt}{(t+\mu)^2} \geqslant \frac{P(\mu)}{2\mu}.$$

Thus $P(\mu)$ is bounded, and so is constant for sufficiently large μ. Hence $p_m = 1$ for all sufficiently large values of m.

It follows that we can write (5.2) as

$$z(\lambda_n) = (n+k+\tfrac{3}{4})\pi + O(1/n), \tag{7.1}$$

where k is a constant.

8. In the next section we shall require the following results. We have

$$z'(\lambda) = \frac{1}{2}\int_0^X \frac{dx}{\{\lambda-q(x)\}^{\frac{1}{2}}} \geqslant \frac{1}{2}\int_0^X \frac{dx}{\{\lambda-q(0)\}^{\frac{1}{2}}} > AX\lambda^{-\frac{1}{2}}. \tag{8.1}$$

Now let $\frac{1}{4}X \leqslant t \leqslant \frac{1}{2}X$. Then

$$z'(\lambda) = \frac{1}{2}\int_0^t \frac{dx}{\{\lambda-q(x)\}^{\frac{1}{2}}} + \frac{\{\lambda-q(t)\}^{\frac{1}{2}}}{q'(t)} - \int_t^X \{\lambda-q(x)\}^{\frac{1}{2}} \frac{q''(x)}{q'^2(x)}\,dx.$$

Hence

$$z''(\lambda) = -\frac{1}{4}\int_0^t \frac{dx}{\{\lambda-q(x)\}^{\frac{3}{2}}} + \frac{1}{2\{\lambda-q(t)\}^{\frac{1}{2}}q'(t)} - \frac{1}{2}\int_t^X \frac{1}{\{\lambda-q(x)\}^{\frac{1}{2}}} \frac{q''(x)}{q'^2(x)}\,dx.$$

Since $q(x)$ is convex downwards,

$$q(\tfrac{1}{2}X) \leqslant \tfrac{1}{2}\{q(0)+q(X)\},$$

and the first term on the right is

$$O\left\{\int_0^{\frac{1}{2}X} \frac{dx}{\{\lambda-q(0)\}^{\frac{3}{2}}}\right\} = O(X\lambda^{-\frac{3}{2}}).$$

The second term is

$$O\left\{\frac{t}{\{\lambda-q(t)\}^{\frac{1}{2}}q(t)}\right\} = O\left\{\frac{t}{\lambda^{\frac{1}{2}}q(t)}\right\} = O\left\{\frac{X}{\lambda^{\frac{1}{2}}q(X)}\right\} = O(X\lambda^{-\frac{3}{2}}).$$

Since $$\frac{q''(x)}{q'^3(x)} = \frac{q''(x)}{q'(x)}\frac{q^2(x)}{q'^2(x)}\frac{1}{q^2(x)} = O\left\{\frac{1}{x}\,x^2\,\frac{1}{q^2(t)}\right\} = O\left(\frac{X}{\lambda^2}\right),$$

the last term is

$$O\left\{\frac{X}{\lambda^2}\int_t^X \frac{q'(x)\,dx}{\{\lambda-q(x)\}^{\frac{1}{2}}}\right\} = O\left\{\frac{X}{\lambda^2}\{\lambda-q(t)\}^{\frac{1}{2}}\right\} = O(X\lambda^{-\frac{3}{2}}).$$

Hence $$z''(\lambda) = O(X\lambda^{-\frac{3}{2}}). \tag{8.2}$$

Also (8.1) gives

$$\int_{\lambda_n}^{\lambda_{n+1}} AX\lambda^{-\frac{1}{2}}\,d\lambda < \int_{\lambda_n}^{\lambda_{n+1}} z'(\lambda)\,d\lambda = z(\lambda_{n+1})-z(\lambda_n) = O(1),$$

so that $$\lambda_{n+1}-\lambda_n = O(X_n^{-1}\lambda_{n+1}^{\frac{1}{2}}). \tag{8.3}$$

It follows that, for n large enough,

$$\lambda_{n+1}-\lambda_n < \epsilon\lambda_{n+1},$$

$$\lambda_{n+1}-\lambda_n < \frac{\epsilon\lambda_n}{1-\epsilon},$$

and so $$\lambda_{n+1}-\lambda_n = o(\lambda_n). \tag{8.4}$$

9. We can now prove that the constant k in (7.1) is zero. Let

$$f(t) = z(t)-\tfrac{1}{2}z(\lambda_n)-\tfrac{1}{2}z(\lambda_{n+1}) \quad (\lambda_n \leqslant t < \lambda_{n+1}).$$

Then, by (7.1), $$f(t) = z(t)-(n+k+\tfrac{5}{4})\pi+o(1).$$

Hence

$$\int_{\lambda_0}^{\infty} \frac{N(t)\,dt}{(t+\mu)^2} = \sum_{n=n_0}^{\infty} \int_{\lambda_n}^{\lambda_{n+1}} \frac{(n+1)\,dt}{(t+\mu)^2}$$

$$= \int_0^{\infty} \left\{\frac{z(t)-f(t)}{\pi}-k-\frac{1}{4}+o(1)\right\} \frac{dt}{(t+\mu)^2}+O\left(\frac{1}{\mu^2}\right)$$

$$= \frac{1}{2}\int_0^{\infty} \frac{dx}{\{\mu+q(x)\}^{\frac{1}{2}}}-\frac{1}{\pi}\int_0^{\infty} \frac{f(t)\,dt}{(t+\mu)^2}-\frac{k+\frac{1}{4}}{\mu}+o\left(\frac{1}{\mu}\right). \tag{9.1}$$

We obtain by integration by parts

$$\int_{\lambda_n}^{\lambda_{n+1}} f(t)\,dt = \tfrac{1}{8}(\lambda_{n+1}-\lambda_n)\{z'(\lambda_n)-z'(\lambda_{n+1})\}-\tfrac{1}{2}\int_{\lambda_n}^{\lambda_{n+1}} (t-\tfrac{1}{2}\lambda_n-\tfrac{1}{2}\lambda_{n+1})^2 z''(t)\,dt$$

$$= O\left\{(\lambda_{n+1}-\lambda_n)^2 \int_{\lambda_n}^{\lambda_{n+1}} X\lambda^{-\frac{3}{2}}\,d\lambda\right\}, \qquad \text{by (8.2)},$$

$$= O[(\lambda_{n+1}-\lambda_n)^2\lambda_n^{-1}\{z(\lambda_{n+1})-z(\lambda_n)\}], \qquad \text{by (8.1)},$$

$$= O\{(\lambda_{n+1}-\lambda_n)^2\lambda_n^{-1}\} = o(\lambda_{n+1}-\lambda_n), \qquad \text{by (8.4)}.$$

Also $f(t)$ is clearly bounded. Hence, if $\lambda_n \leqslant T < \lambda_{n+1}$,

$$F(T) = \int_0^T f(t)\,dt = o\Big\{\sum_{\nu \leqslant n}(\lambda_\nu - \lambda_{\nu+1})\Big\} + O(\lambda_{n+1} - \lambda_n)$$

$$= o(\lambda_n) = o(T).$$

Hence
$$\int_0^\infty \frac{f(t)\,dt}{(t+\mu)^2} = 2\int_0^\infty \frac{F(t)\,dt}{(t+\mu)^3} = o\left\{\int_0^\infty \frac{t\,dt}{(t+\mu)^3}\right\} = o\left(\frac{1}{\mu}\right). \tag{9.2}$$

From this result, (9.1), and (6.7), it follows that $k = 0$, so that finally

$$z(\lambda_n) = (n+\tfrac{3}{4})\pi + O(1/n). \tag{9.3}$$

10. The above argument fails if (6.1) is divergent. In any case we can operate similarly with the formula

$$\int_0^\infty \int_0^\infty G^2(x, y, -\mu)\,dxdy = \sum_{n=0}^\infty \frac{1}{(\lambda_n+\mu)^2},$$

the left-hand side being

$$O\left\{\int_0^\infty \frac{e^{-2i\xi(x)}\,dx}{\{\mu+q(x)\}^{\frac{1}{2}}} \int_x^\infty \frac{e^{2i\xi(y)}\,dy}{\{\mu+q(y)\}^{\frac{1}{2}}}\right\} = O\left\{\int_0^\infty \frac{dx}{\{\mu+q(x)\}^{\frac{3}{2}}}\right\},$$

which is convergent since $q(x) > Ax$. A more complicated argument of the above type then leads to the same result.

11. Other problems

In the corresponding problem in which the boundary condition is $\psi'(0) = 0$, (4.8) leads to

$$z(\lambda_n) = (m+\tfrac{1}{4})\pi + O(1/m) \tag{11.1}$$

instead of (5.2). It can then be shown as in § 9 that $m = n$; for we have a formula similar to (6.2) with $\cos\xi(x)$ instead of $\sin\xi(x)$, and so a formula similar to (6.3) with $\frac{1}{4}\mu^{-1}$ instead of $-\frac{1}{4}\mu^{-1}$. This gives the result stated. Actually (11.1), with $m = n$, holds in the case of any boundary condition

$$\psi(0)\cos\alpha + \psi'(0)\sin\alpha = 0$$

provided that $\sin\alpha \neq 0$.

Consider finally the corresponding problem in which the interval is $(-\infty, \infty)$, $q(x)$ satisfying similar conditions at both ends. Let the function

corresponding to negative values of x in the same way as $\psi(x, y)$ corresponds to positive values of x be denoted by $\psi_1(x, \lambda)$, and let

$$z_1(\lambda) = \int_{X}^{0} \{\lambda - q(t)\}^{\frac{1}{2}}\, dt.$$

Then

$$\psi_1(0, \lambda) = 2e^{-\frac{3}{4}i\pi}\{\lambda - q(0)\}^{-\frac{1}{4}}\{\cos(z_1 - \tfrac{1}{4}\pi) + O(z_1^{-1})\},$$

$$\psi_1'(0, \lambda) = -2e^{-\frac{3}{4}i\pi}\{\lambda - q(0)\}^{\frac{1}{4}}\{\sin(z_1 + \tfrac{1}{4}\pi) + O(z_1^{-1})\}.$$

The eigenvalues are the zeros of the Wronskian of ψ_1 and ψ, and so satisfy an equation of the form

$$\cos\{z_1(\lambda) - \tfrac{1}{4}\pi\}\sin\{z(\lambda) - \tfrac{1}{4}\pi\} + \sin\{z_1(\lambda) - \tfrac{1}{4}\pi\}\cos\{z(\lambda) - \tfrac{1}{4}\pi\}$$
$$= O\{1/z_1(\lambda) + 1/z(\lambda)\},$$

i.e.

$$\cos\{z_1(\lambda) + z(\lambda)\} = O\{1/z_1(\lambda) + 1/z(\lambda)\}.$$

Hence

$$z_1(\lambda_n) + z(\lambda_n) = (m + \tfrac{1}{2})\pi + O\{1/z_1(\lambda_n) + 1/z(\lambda_n)\}.$$

In this case the equation corresponding to (6.3) is

$$\int_{-\infty}^{\infty} G(x, x, -\mu)\, dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{\{\mu + q(x)\}^{\frac{1}{2}}} + o\left(\frac{1}{\mu}\right),$$

and so $m = n$. This proves (1.2) with $\lambda = \lambda_n$.

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SOME ASYMPTOTIC FORMULAE IN THE THEORY OF PARTITIONS

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1. Introduction

LET u_k ($k = 1, 2, 3, \ldots$) be a sequence of positive integers which is monotonic increasing for $k \geqslant k_0$ and let $p_u(n)$ denote the number of partitions of n into distinct members of the sequence u_k, i.e. the number of solutions of

$$n = u_{\nu_1} + u_{\nu_2} + \ldots + u_{\nu_r} \quad (1 \leqslant \nu_1 < \ldots < \nu_r),$$

r being arbitrary. Our aim is to investigate the behaviour of the function $p_u(n)$ for sequences u_k of a fairly general nature. We shall in fact obtain an asymptotic expansion [see Theorem 2, § 2] for $p_u(n)$ for sequences u_k satisfying the following two conditions (I) and (II) below:†

$$s = \lim_{k \to \infty} \frac{\log u_k}{\log k} \text{ exists;} \tag{I}$$

$$J_k = \inf\Big\{(\log k)^{-1} \sum_{\nu=1}^{k} \|u_\nu \alpha\|^2\Big\} \to \infty \text{ as } k \to \infty, \tag{II}$$

where the lower bound is taken over those α satisfying

$$\tfrac{1}{2} u_k^{-1} < \alpha \leqslant \tfrac{1}{2}.$$

It will be shown in § 4 that, among many others, the following sequences satisfy conditions (I) and (II):

(i) $u_k = p_k$, where p_k is the kth prime number;

(ii) $u_k = f(k)$, where $f(x)$ is a polynomial which takes only integral values for integral x and has the property that corresponding to every prime p there exists an integer x such that $p \nmid f(x)$.

(iii) $u_k = f(p_k)$, where $f(x)$ is a polynomial which takes only integral values for integral x and has the property that corresponding to every prime p there exists an x such that $p \nmid xf(x)$.

† $\|\theta\|$ denotes the distance of θ from the nearest integer.

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In Case (i) one can obtain from the asymptotic expansion (1) for $p_u(n)$, by means of a routine calculation, the formula

$$\log p_u(n) = (\tfrac{2}{3})^{\frac{1}{2}}\pi\left(\frac{n}{\log n}\right)^{\frac{1}{2}}\left\{1+O\left(\frac{\log\log n}{\log n}\right)\right\}.$$

In Case (ii), if $\qquad u_k = f(k) = ak^s+bk^{s-1}+\ldots,$

one can deduce from (1) that

$$p_u(n) = \{1+O(n^{-1/(s+1)})\}2^{-(1+b/sa)}\left\{\frac{s}{\pi(s+1)}\right\}^{\frac{1}{2}}\frac{1}{n}\left(\frac{\gamma^s n}{a}\right)^{1/2(s+1)}\times$$

$$\times\exp\left\{(s+1)\left(\frac{\gamma^s n}{a}\right)^{1/(s+1)}\right\},$$

where $\qquad \gamma = (1-2^{-1/s})\frac{1}{s}\Gamma\left(1+\frac{1}{s}\right)\zeta\left(1+\frac{1}{s}\right).$

These calculations which, although straightforward, are rather lengthy, are omitted here. Asymptotic expansions for $p_u(n)$ appear to have been obtained previously only in the special case when $f(x) = x^s$.

Ingham† has shown how the main term of an asymptotic formula can be obtained, under certain conditions, from a Tauberian theorem, but the conditions are much more restrictive than those imposed here.

A problem closely allied to the one considered in the present paper is that of investigating the behaviour of the function $p_u^*(n)$, where $p_u^*(n)$ denotes the number of partitions of n into members of the sequence u_k, without the restriction that these members should be distinct. The methods of this paper can also be applied to finding an asymptotic formula for $p_u^*(n)$ and this will be discussed briefly in § 5. But, although condition (II) can be considerably relaxed, the results are comparatively weak in the sense that much more is probably true in this case.

Bateman and Erdős‡ have recently established necessary and sufficient conditions on the sequence u_k for the function $p_u^*(n)$ to be monotonic increasing (without obtaining an asymptotic formula). Dr. Erdős has proposed to us the problem of proving that $p_u(n)$ is monotonic for $n \geqslant n_0$ in the special case when u_k is the sequence of prime numbers. We shall in fact prove this for all sequences u_k satisfying conditions (I) and (II).

† A. E. Ingham, 'A Tauberian theorem for partitions', *Ann. Math. Princeton* 42 (1941) 1075–90.

‡ In a paper to appear in *Mathematika*.

More precisely, we shall show in § 2 that the asymptotic formula (1) satisfied by $p_u(n)$ for such sequences implies

THEOREM 1. *There exists an n_0 and a positive constant c, depending only on δ and the sequence, such that for all $n > n_0$,*

$$p_u(n+1)-p_u(n) > cn^{-s/(s+1)-\delta}p_u(n),$$

for any positive δ.

The following conventions and notations will be used throughout the paper.

$\sum_1^r{}_u f(u)$ is used as an abbreviation for $\sum_{\nu=1}^{r} f(u_\nu)$.

The letter c denotes a positive number which is independent of n but may depend on δ, m, and the sequence u_ν. We shall usually refrain from using suffixes with these 'constants' c except where several such c appear in one formula (so that c does not always refer to the same constant). Except where otherwise stated, the constants implied by the O and o notations are of the same nature as the constants c.

The letter δ denotes an arbitrarily small positive number; n denotes a sufficiently large integer.

In §§ 2 and 3 we assume that the sequence u_k satisfies conditions (I) and (II).

2. In this section we deduce Theorem 1 from the asymptotic formula for $p_u(n)$ which will be proved in § 3. This asymptotic formula may be stated in the form of the following theorem.

THEOREM 2. *For any fixed positive integer m (> 1),*

$$p_u(n) = (2\pi A_2)^{-\frac{1}{2}}\exp\left[\sum_1^\infty{}_u \left\{\frac{\eta u}{e^{\eta u}+1}+\log(1+e^{-\eta u})\right\}\right]\times$$

$$\times\left\{1+\sum_{\rho=1}^{m-2} D_\rho + O(n^{-(m-1)(s+1)^{-1}+\delta})\right\}, \quad (1)$$

where $\eta = \eta(n)$ is determined from

$$n = \sum_1^\infty{}_u \frac{u}{e^{\eta u}+1}, \quad (2)$$

and D_ρ $(\rho = 1, 2, \ldots, m-1)$ is given by

$$D_\rho = A_2^{-6\rho} \sum_{\mu_1=2}^{\infty} \cdots \sum_{\mu_{5\rho}=2}^{\infty} d_{\mu_1,\mu_2,\ldots,\mu_{5\rho}} A_{\mu_1} A_{\mu_2} \ldots A_{\mu_{5\rho}} \quad (3)$$

the summation being subject to

$$\mu_1+\mu_2+\ldots+\mu_{5\rho} = 12\rho,$$

where the d's are certain numerical coefficients, and

$$A_\mu = A_\mu(n) = \sum_1^\infty{}_u\, u^\mu g_\mu(e^{\eta u})(e^{\eta u}+1)^{-\mu}. \tag{4}$$

Here $g_\mu(x)$ is a certain polynomial of degree less than μ, and, in particular, $g_1(x) = 1$ and $g_2(x) = x$.

In order to obtain estimates for η and the A's we shall require the following lemma.

LEMMA 1. *If η is determined by (2) and μ, ν are integers satisfying $0 \leqslant \nu < \mu$, then*

$$\lim_{n\to\infty} \frac{\log \eta}{\log n} = -\frac{s}{1+s}; \tag{5}$$

and, if

$$A_{\mu\nu} = \sum_1^\infty{}_u \frac{u^\mu e^{\eta\nu u}}{(e^{\eta u}+1)^\mu},$$

then

$$\lim_{n\to\infty} \frac{\log A_{\mu\nu}}{\log n} = \frac{\mu s+1}{s+1}. \tag{6}$$

Proof. Let δ (as always) be a small positive number. By condition (I) there exists $k_0 = k_0(\delta)$ such that

$$k^{s+\delta} > u_k > k^{s-\delta} \quad \text{for } k > k_0.$$

Hence, noting that $\eta \to 0$ as $n \to \infty$, and using the substitution $\eta x^{s-\delta} = y$, we obtain

$$A_{\mu\nu} < \sum_1^\infty{}_u\, u^\mu e^{-\eta u} < c + \sum_{k=1}^\infty k^{\mu(s+\delta)} e^{-\eta k^{s-\delta}}$$

$$< c + \int_1^\infty (x+1)^{\mu(s+\delta)} e^{-\eta x^{s-\delta}}\, dx < c_1 + c_2 \int_0^\infty x^{\mu(s+\delta)} e^{-\eta x^{s-\delta}}\, dx$$

$$< c_1 + c_2\, \eta^{-(1+\mu s+\delta\mu)(s-\delta)^{-1}} \int_0^\infty y^{(1+\mu s+\mu\delta)(s-\delta)^{-1}-1} e^{-y}\, dy.$$

Hence we have $A_{\mu\nu} < c_1 + c_2\, \eta^{-(1+\mu s+\mu\delta)(s-\delta)^{-1}}$.

On the other hand,

$$A_{\mu\nu} \geqslant \sum_1^\infty{}_v\, u^\mu (e^{\eta u}+1)^{-\mu} > -c + \sum_{k=2}^\infty k^{\mu(s-\delta)} e^{-\eta k^{s+\delta}}$$

$$> -c + \int_2^\infty (x-1)^{\mu(s-\delta)} e^{-\eta x^{s+\delta}}\, dx > -c_1 + c_2\, \eta^{-(1+\mu s-\mu\delta)(s+\delta)^{-1}}.$$

Letting $n \to \infty$ (so that $\eta \to 0$) in these inequalities, we obtain

$$\liminf \frac{\log A_{\mu\nu}}{(-\log \eta)} \geqslant (1+\mu s-\mu\delta)(s+\delta)^{-1},$$

$$\limsup \frac{\log A_{\mu\nu}}{(-\log \eta)} \leqslant (1+\mu s+\mu\delta)(s-\delta)^{-1}.$$

Since this is true for every positive $\delta \leqslant \frac{1}{2}s$, we must have

$$\lim_{n\to\infty} \frac{\log A_{\mu\nu}}{\log \eta} = -(\mu+s^{-1}). \tag{7}$$

In particular, since $n = A_{10}$ by (2), we obtain

$$\lim_{n\to\infty} \frac{\log n}{\log \eta} = -(1+s^{-1}) \tag{8}$$

on writing $\mu = 1$, $\nu = 0$ in (7). This proves (5), which in conjunction with (7), implies (6).

Lemma 1 has the following obvious corollary.

COROLLARY TO LEMMA 1. *If the A_μ are defined as in Theorem 2 [see (4)], then*

$$\lim_{n\to\infty} \frac{\log A_2}{\log n} = \frac{2s+1}{s+1}, \quad \limsup_{n\to\infty} \frac{\log A_\mu}{\log n} \leqslant \frac{\mu s+1}{s+1} \quad (\mu \geqslant 3).$$

We are now in a position to obtain estimates of the D_ρ defined by (3). In fact, we have by the above corollary, for $\rho = 1, 2, \ldots, m-2$,

$$\limsup_{n\to\infty} \frac{\log D_\rho}{\log n} \leqslant \frac{-6\rho(2s+1)+(\mu_1+\ldots+\mu_{5\rho})s+5\rho}{s+1} = -\frac{\rho}{s+1}. \tag{9}$$

We now prove Theorem 1. Let $\Delta\eta$, ΔA_μ be the increments in η, A_μ respectively when n is replaced by $n+1$. By the definition (2) of η, we obtain, on using Taylor's theorem and Lemma 1,

$$\begin{aligned} 1 &= \sum_1^\infty u \left\{ \frac{u}{e^{u(\eta+\Delta\eta)}+1} - \frac{u}{e^{\eta u}+1} \right\} \\ &= \sum_1^\infty u \left[\frac{-u^2(\Delta\eta)e^{\eta u}}{(e^{\eta u}+1)^2} + O\left\{ \frac{u^3(\Delta\eta)^2(1+e^{2\eta u})}{(e^{\eta u}+1)^3} \right\} \right] \\ &= -A_2\Delta\eta + O\{n^{(3s+1)(s+1)^{-1}+\delta}(\Delta\eta)^2\}. \end{aligned}$$

Again, using Lemma 1 Corollary, it follows that

$$\Delta\eta = -A_2^{-1} + O(A_2^{-3}\, n^{(3s+1)(s+1)^{-1}+\delta}) = -A_2^{-1} + O(n^{-(3s+2)(s+1)^{-1}+2\delta}).$$

Further, by a similar argument,

$$\Delta A_\mu = O(n^{\mu s(s+1)^{-1}+1+\delta}\Delta\eta) = O(n^{(\mu-1)s(s+1)^{-1}+\delta}) \quad (\mu = 2, 3,\ldots), \tag{10}$$

and hence

$$\Delta D_\rho = O(n^{-(\rho-s)(s+1)^{-1}+\delta}\Delta\eta) = O(n^{-\rho(s+1)^{-1}-1+\delta}). \tag{11}$$

Choosing $m > s+2$ in (1), we obtain from (11),

$$\Delta\Big(1+\sum_{\rho=1}^{m-1} D_\rho + O(n^{-(m-1)(s+1)^{-1}+\delta})\Big) = O(n^{-1+\delta}).$$

Further, $$-\tfrac{1}{2}\Delta\log A_2 = -\tfrac{1}{2}\log\Big(1+\frac{\Delta A_2}{A_2}\Big) = O(n^{-1+\delta}).$$

Finally,

$$\begin{aligned}
&\Delta\Big\{\sum_1^\infty{}_u \Big(\frac{\eta u}{e^{\eta u}+1}+\log(1+e^{-\eta u})\Big)\Big\} \\
&\quad = \sum_1^\infty{}_u \Big(\frac{u}{e^{\eta u}+1} - \frac{\eta u^2 e^{\eta u}}{(e^{\eta u}+1)^2} - \frac{u}{e^{\eta u}+1}\Big)\Delta\eta + O\{n^{(2s+1)(s+1)^{-1}+\delta}(\Delta\eta)^2\} \\
&\quad = -\eta A_2\Delta\eta + O(n^{-(2s+1)(s+1)^{-1}+2\delta}) \\
&\quad = \eta + O(n^{-1+\delta}).
\end{aligned}$$

Thus formula (1) gives

$$\log p_u(n+1) - \log p_u(n) = \Delta\log p_u(n) = \eta + O(n^{-1+\delta}),$$

which, in view of (5), proves Theorem 1.

3. In this section we shall prove the asymptotic formula for $p_u(n)$ which was stated in the preceding section (Theorem 2). We define a generating function corresponding to the definition of $p_u(n)$, namely

$$F(w) = \prod_1^\infty{}_u (1+w^u) = \sum_{n=0}^\infty p_u(n) w^n. \tag{12}$$

Using Cauchy's theorem to express $p_u(n)$ as a contour integral, we obtain

$$p_u(n) = \frac{1}{2\pi i}\int F(w) w^{-(n+1)}\, dw, \tag{13}$$

the path of integration being a circle with centre at the origin and radius less than 1. For reasons which have been explained in a previous paper,† we choose the radius of this circle to satisfy

$$\rho\frac{F'(\rho)}{F(\rho)} = n.$$

† G. Szekeres, 'An asymptotic formula in the theory of partitions (II)', *Quart. J. of Math.* (Oxford) (2) 4 (1953) 96–111.

Writing $\rho = e^{-\eta}$, where $\eta > 0$, we see that

$$\rho \frac{F'(\rho)}{F(\rho)} = \sum_1^\infty {}_u \frac{u\rho^u}{1+\rho^u} = \sum_1^\infty {}_u \frac{u}{e^{\eta u}+1},$$

so that η is in fact defined by (2).

Writing $w = e^{-\eta+i\theta}$ in (13) and using (2), we obtain

$$p_u(n) = \frac{1}{2\pi i} \int_{-\pi}^{\pi} \left\{ \prod_1^\infty {}_u (1+e^{-u\eta+iu\theta}) \right\} e^{(n+1)\eta-(n+1)i\theta} ie^{-\eta+i\theta}\, d\theta$$

$$= \frac{1}{2\pi} \exp\left\{ \eta n + \sum_1^\infty {}_u \log(1+e^{-\eta u}) \right\} \times$$

$$\times \int_{-\pi}^{\pi} \exp\left\{ -in\theta + \sum_1^\infty {}_u \log \frac{1+e^{-\eta u+i\theta u}}{1+e^{-\eta u}} \right\} d\theta$$

$$= \frac{1}{2\pi} \exp\left\{ \sum_1^\infty {}_u \left(\frac{\eta u}{e^{\eta u}+1} + \log(1+e^{-\eta u}) \right) \right\} (I_1+I_2+I_3) \text{ (say)}, \quad (14)$$

where the last integral appearing above has been split up into the integrals

$$I_1 = \int_{-\theta_0}^{\theta_0}, \qquad I_2 = \int_{\theta_0}^{\pi}, \qquad I_3 = \int_{-\pi}^{-\theta_0}.$$

The value of θ_0 will be chosen later, subject to the condition

$$\theta_0 = o(\eta).$$

In order to estimate the integral I_1, we expand the integrand as a power series in θ (apart from one factor, as will be seen later). Now, for any u,

$$L(u) = \log \frac{1+e^{-\eta u+i\theta u}}{1+e^{-\eta u}} = \log\left(1 + \frac{e^{i\theta u}-1}{e^{\eta u}+1}\right)$$

$$= \sum_{l=1}^\infty (-1)^{l-1} \frac{1}{l} (1+e^{\eta u})^{-l} \left\{ \sum_{k=1}^\infty (i\theta u)^k \right\}^l.$$

The multiple sum is absolutely convergent for $|\theta| \leqslant \theta_0$, provided that $\theta_0 \leqslant \frac{1}{2}\eta$, since in this case

$$\sum_{l=1}^\infty \frac{1}{l} (1+e^{\eta u})^{-l} \left\{ \sum_{k=1}^\infty \frac{1}{k!} |\theta u|^k \right\}^l$$

is the formal expansion of

$$-\log\left\{1-\frac{e^{\theta u}-1}{e^{\eta u}+1}\right\}$$

and
$$\frac{e^{|\theta u|}-1}{e^{\eta u}+1} \leqslant \frac{e^{\frac{1}{2}\eta u}-1}{e^{\eta u}+1} < \frac{e^{\frac{1}{2}\eta u}-1}{e^{\eta u}-1} = \frac{1}{e^{\frac{1}{2}\eta u}+1} < \frac{1}{2}.$$

Hence we may write

$$\sum_{l=1}^{\infty}(-1)^{l-1}\frac{1}{l}(e^{\eta u}+1)^{-l}\left\{\sum_{k=1}^{\infty}\frac{1}{k!}(i\theta u)^k\right\}^l = \sum_{\nu=1}^{\infty}\frac{a_\nu}{\nu!}(i\theta u)^\nu,$$

where
$$a_\nu = \sum_{l=1}^{\nu}\sum_{\substack{k_1=1\\ k_1+\ldots+k_l=\nu}}^{\infty}\cdots\sum_{k_l=1}^{\infty}\frac{(-1)^{l-1}}{l}(e^{\eta u}+1)^{-l}\frac{\nu!}{k_1!\ldots k_l!},$$

so that
$$a_\nu = (e^{\eta u}+1)^{-\nu}g_\nu(e^{\eta u}),$$

where $g_\nu(x)$ is a polynomial of degree not exceeding $\nu-1$ and in particular

$$g_1(x) = 1, \qquad g_2(x) = x.$$

Now, for $|z| \leqslant \theta_0 u < \frac{1}{2}\eta u$,

$$\log\left(\frac{1+e^{-\eta u+z}}{1+e^{-\eta u}}\right) = \sum_{l=1}^{\infty}(-1)^{l-1}\frac{1}{l}(e^{\eta u}+1)^{-l}(e^z-1)^l$$

$$= \sum_{\nu=1}^{\infty}a_\nu z^\nu.$$

Write, for a fixed integer $m \geqslant 2$,

$$\sum_{l=1}^{2m-1}(-1)^{l-1}\frac{1}{l}(e^{\eta u}+1)^{-l}(e^z-1)^l = \sum_{\nu=1}^{\infty}\frac{1}{\nu!}a_\nu^{(1)}z^\nu,$$

$$\sum_{l=2m}^{\infty}(-1)^{l-1}\frac{1}{l}(e^{\eta u}+1)^{-l}(e^z-1)^l = \sum_{\nu=1}^{\infty}\frac{1}{\nu!}a_\nu^{(2)}z^\nu,$$

so that
$$a_\nu^{(1)}+a_\nu^{(2)} = a_\nu.$$

Now, since the coefficients in the expansion of $(e^z-1)^l$ do not exceed the corresponding coefficients in the expansion of e^{zl},

$$\begin{aligned}\Big|\sum_{\nu=2m}^{\infty}\frac{1}{\nu!}a_\nu^{(1)}z^\nu\Big| &\leqslant \sum_{l=1}^{2m-1}\frac{1}{l}(e^{\eta u}+1)^{-l}\sum_{k=2m}^{\infty}\frac{1}{k!}l^k|z|^k\\ &< \sum_{l=1}^{2m-1}(e^{\eta u}+1)^{-l}(c|z|^{2m}e^{|z|})\\ &< c\frac{|z|^{2m}e^{\theta_0 u}}{e^{\eta u}+1}\\ &< c\frac{|z|^{2m}}{e^{\frac{1}{2}\eta u}+1},\end{aligned}$$

and $$\Big|\sum_{\nu=2m}^{\infty}\frac{1}{\nu!}a_\nu^{(2)}z^\nu\Big| < \sum_{l=2m}^{\infty}\left(\frac{e^{|z|}-1}{e^{\eta u}+1}\right)^l < c\frac{|z|^{2m}e^{2m\theta_0 u}}{(e^{\eta u}+1)^{2m}} < c\frac{|z|^{2m}}{e^{\frac{1}{2}\eta u}+1}.$$

Thus $$\log\left(\frac{1+e^{-\eta u+z}}{1+e^{-\eta u}}\right) = \sum_{\nu=1}^{2m-1}\frac{1}{\nu!}a_\nu z^\nu + O\{|z|^{2m}(e^{\frac{1}{2}\eta u}+1)^{-1}\}.$$

Collecting our results, we obtain

$$L(u) = \sum_{\nu=1}^{2m-1}\frac{1}{\nu!}(e^{\eta u}+1)^{-\nu}g_\nu(e^{\eta u})(i\theta u)^\nu + O\{(\theta u)^{2m}(e^{\frac{1}{2}\eta u}+1)^{-1}\},$$

and hence

$$\begin{aligned}I_1 &= \int_{-\theta_0}^{\theta_0}\exp\Big\{-in\theta+\sum_1^\infty{}_u\, L(u)\Big\}\,d\theta\\ &= \int_{-\theta_0}^{\theta_0}\exp\Big\{-in\theta+\sum_{\mu=1}^{2m-1}\frac{1}{\mu!}\sum_1^\infty{}_u\,(e^{\eta u}+1)^{-\mu}g_\mu(e^{\eta u})(i\theta u)^\mu+\\ &\qquad\qquad +O\Big(\sum_1^\infty{}_u\,(\theta u)^{2m}(e^{\frac{1}{2}\eta u}+1)^{-1}\Big)\Big\}\,d\theta.\end{aligned}$$

Since $g_1(x)=1$, we have, by (2),

$$-in\theta+\sum_1^\infty{}_u\,(e^{\eta u}+1)^{-1}g_1(e^{\eta u})(i\theta u) = 0,$$

and hence, recalling that $g_2(x)=x$, we obtain

$$I_1 = \int_{-\theta_0}^{\theta_0}\exp\Big\{-\tfrac{1}{2}A_2\theta^2+\sum_{\mu=3}^{2m-1}\frac{1}{\mu!}A_\mu(i\theta)^\mu+O\Big(|\theta|^{2m}\sum_1^\infty{}_u\,u^{2m}e^{-\frac{1}{2}\eta u}\Big)\Big\}\,d\theta; \tag{15}$$

where the A_μ are defined by (4).

We now choose θ_0 by writing

$$\theta_0 = \eta^{\frac{1}{5}}(\tfrac{1}{2}A_2)^{-\frac{2}{5}} = \eta B^{-\frac{2}{5}}, \quad \text{where } B = \tfrac{1}{2}\eta^2 A_2. \tag{16}$$

By (9) and the corollary to Lemma 1, we have

$$\lim_{n\to\infty} \frac{\log B}{\log n} = \frac{1}{s+1}, \tag{17}$$

and we see that θ_0 does in fact satisfy $\theta_0 = o(\eta)$. Further, for $\mu > 2$, we have by the corollary to Lemma 1,

$$\limsup \frac{\log(A_\mu \theta_0^\mu)}{\log n} \leqslant \limsup \frac{\log \eta^{\frac{1}{5}\mu} A_2^{-\frac{2}{5}\mu} A_\mu}{\log n}$$

$$\leqslant -\frac{s}{1+s}\frac{\mu}{5} - \frac{2s+1}{s+1}\frac{2\mu}{5} + \frac{\mu s+1}{s+1} = -\left(\frac{2\mu}{5}-1\right)(s+1)^{-1} \leqslant -\tfrac{1}{5}(s+1)^{-1},$$

so that, for any $\delta > 0$,

$$A_\mu \theta_0^\mu = O(n^{-\frac{1}{5}(s+1)^{-1}+\delta}). \tag{18}$$

Also it follows from the first part of the proof of Lemma 1 (with η replaced by $\frac{1}{2}\eta$) that

$$\limsup \frac{\sum_1^\infty{}_u\, u^{2m} e^{-\frac{1}{2}\eta u}}{\log n} \leqslant \frac{2ms+1}{s+1}, \tag{19}$$

so that, by the same calculation as was used to prove (18) with $\mu = 2m$ (the right-hand side of (19) is also the upper limit of $(\log A_{2m})/(\log n)$), we have

$$\theta_0^{2m} \sum_1^\infty{}_u\, u^{2m} e^{-\frac{1}{2}\eta u} = O(n^{-\frac{1}{5}(s+1)^{-1}+\delta}). \tag{20}$$

We now write

$$A'_\mu = A_\mu \quad (\mu = 2,\ldots, 2m-1), \qquad A'_{2m} = \sum_1^\infty{}_u\, u^{2m} e^{-\frac{1}{2}\eta u}.$$

We note that the inequality (19) satisfied by A'_{2m} corresponds to the inequality for A_{2m} given in Lemma 1 Corollary. Then, by (18) and (20), we obtain for $|\theta| \leqslant \theta_0$,

$$\exp\left(\sum_{\mu=3}^{2m-1} \frac{1}{\mu!} A_\mu (i\theta)^\mu + O(A'_{2m}|\theta|^{2m})\right)$$

$$= 1 + \sum_{\nu=1}^{5m} \frac{1}{\nu!}\left\{\sum_{\mu=3}^{2m-1} \frac{1}{\mu!} A_\mu (i\theta)^\mu + O(A'_{2m}|\theta|^{2m})\right\}^\nu + O(n^{-m/(s+1)})$$

$$= 1 + E(\theta) + O\{R(|\theta|)\} + O(n^{-m/(s+1)}),$$

where

$$E(\theta) = \sum_{\nu=1}^{5m} \sum_{\mu_1=3}^{2m-1} \cdots \sum_{\mu_\nu=3}^{2m-1} \frac{1}{\nu!} \frac{1}{\mu_1! \ldots \mu_\nu!} A_{\mu_1} \ldots A_{\mu_\nu} (i\theta)^{\mu_1+\ldots+\mu_\nu}$$

and

$$R(|\theta|) = \sum_{\nu=1}^{5m} \sum_{\mu_1=3}^{2m} \cdots \sum_{\mu_\nu=3}^{2m} A'_{\mu_1} \ldots A'_{\mu_\nu} |\theta|^{\mu_1+\ldots+\mu_\nu},$$

the last summation being subject to $\max(\mu_1, \ldots, \mu_\nu) = 2m$.

Now, using (15), (16) and making the substitution $t = (\frac{1}{2}A_2)^{\frac{1}{2}}\theta$, we obtain

$$I_1 = \int_{-\theta_0}^{\theta_0} e^{-\frac{1}{2}A_2\theta^2}\left[1 + E(\theta) + O\{R(|\theta|)\} + O(n^{-m(s+1)^{-1}})\right] d\theta$$

$$= (\tfrac{1}{2}A_2)^{-\frac{1}{2}} \int_{-B^{1/10}}^{B^{1/10}} e^{-t^2}\left[1 + E\{t(\tfrac{1}{2}A_2)^{-\frac{1}{2}}\} + O\{R(|t|A_2^{-\frac{1}{2}})\} + O(n^{-m(s+1)^{-1}})\right] dt.$$

Now for each possible ρ, σ we group together those terms in the multiple sum defining $E\{t(\frac{1}{2}A_2)^{-\frac{1}{2}}\}$ for which

$$\mu_1 + \ldots + \mu_\nu = \sigma \quad \text{and} \quad \rho = \sigma - 2\nu. \tag{21}$$

Since in the multiple sum defining E each μ is at least 3, (21) implies $3\nu \leqslant \sigma = \rho + 2\nu$, and hence $\nu \leqslant \rho$ and $\sigma \leqslant 3\rho$. Further, (21) implies

$$\rho < \sigma < \nu(2m) \leqslant 10m^2.$$

Thus we obtain

$$E\{t(\tfrac{1}{2}A_2)^{-\frac{1}{2}}\} = \sum_{\rho=1}^{10m^2} \sum_{\sigma=3}^{3\rho} C_{\rho\sigma}(it)^\sigma,$$

where

$$C_{\rho\sigma} = \sum_{\nu=1}^{5m} \sum_{\mu_1=3}^{2m-1} \cdots \sum_{\mu_\nu=3}^{2m-1} \frac{1}{\nu!\,\mu_1! \ldots \mu_\nu!} A_{\mu_1} \ldots A_{\mu_\nu} (\tfrac{1}{2}A_2)^{-\frac{1}{2}\sigma},$$

subject to

$$\mu_1 + \ldots + \mu_\nu = \sigma, \qquad \rho = \sigma - 2\nu.$$

We group together the terms of the multiple sum $R(|\theta|A_2^{-\frac{1}{2}})$ in exactly the same way, obtaining

$$R(|t|A_2^{-\frac{1}{2}}) = \sum_{\rho=2m-2}^{10m^2} \sum_{\sigma=3}^{3\rho} C'_{\rho\sigma} |t|^\sigma,$$

since in the multiple sum defining R at least one of the μ is equal to $2m$ and the other μ (if any) are not less than 3, so that (21) implies

$$\rho + 2\nu = \sigma \geqslant 2m + 3(\nu - 1)$$

and hence $\rho \geqslant 2m - 2$.

Now by Lemma 1 Corollary,

$$C_{\rho\sigma} = O(n^{(s+1)^{-1}\{(\sigma s+\nu)+(2s+1)(-\frac{1}{2}\sigma)+\delta\}}) = O(n^{-\frac{1}{2}\rho(s+1)+\delta}).$$

Similarly (since the inequalities satisfied by the A'_μ correspond to those satisfied by the A_μ),

$$C'_{\rho\sigma} = O(n^{-\frac{1}{2}\rho(s+1)+\delta}).$$

Thus, collecting together the above results, we have

$$I_1 = (\tfrac{1}{2}A_2)^{-\frac{1}{2}} \int_{-B^{1/10}}^{B^{1/10}} e^{-t^2}\Big\{1 + \sum_{\rho=1}^{2m-3}\sum_{\sigma=3}^{3\rho} C_{\rho\sigma}(it)^\sigma + O\Big(n^{-(m-1)(s+1)^{-1}+\delta}\sum_{\sigma=0}^{10m^2}|t|^\sigma\Big)\Big\}\,dt.$$

Now, by (17), we have, for $0 \leqslant \sigma \leqslant 10m^2$,

$$\int_{B^{1/10}}^{\infty} e^{-t^2}t^\sigma\,dt = O(e^{-n^\epsilon}),$$

so that, since, for odd σ, $t^\sigma e^{-t^2}$ is an odd function of t, we obtain on writing $\sigma = 2\sigma'$ and $\rho = 2\rho'$ for even σ,

$$I_1 = (\tfrac{1}{2}A_2)^{-\frac{1}{2}}\Big[\int_{-\infty}^{\infty} e^{-t^2}\Big\{1 + \sum_{\rho'=1}^{m-2}\sum_{\sigma'=3}^{3\rho'}(-1)^{\sigma'}C_{2\rho',2\sigma'}t^{2\sigma'}\Big\}\,dt + O(n^{-(m-1)(s+1)^{-1}+\delta})\Big]$$

$$= (2\pi/A_2)^{\frac{1}{2}}\Big\{1 + \sum_{\rho=1}^{m-2} D_\rho + O(n^{-(m-1)(s+1)^{-1}+\delta})\Big\} \tag{22}$$

where, for certain numerical constants $e_{2\sigma}$,

$$D_\rho = \sum_{\sigma=3}^{3\rho} e_{2\sigma}C_{2\rho,2\sigma}.$$

We shall now show that the D_ρ have the form (3) stated in the introduction. We have

$$C_{2\rho,2\sigma} = A_2^{-6\rho}2^\sigma \sum_{\nu=1}^{5m}\sum_{\mu_1=3}^{2m-1}\cdots\sum_{\mu_\nu=3}^{2m-1}\frac{1}{\nu!\,\mu_1!\ldots\mu_\nu!}A_{\mu_1}\ldots A_{\mu_\nu}A_2^{5\rho-\nu},$$

subject to $$\mu_1+\ldots+\mu_\nu = 2\rho+2\nu.$$

Noting that $2\rho \geqslant \nu$ for each term of this multiple sum, and writing

$$A_2^{5\rho-\nu} = \sum_{\mu_{\nu+1}=2}^{2}\cdots\sum_{\mu_{5\rho}=2}^{2} A_{\mu_{\nu+1}}\ldots A_{\mu_{5\rho}},$$

we obtain

$$C_{2\rho,2\sigma} = 2^\sigma A_2^{-6\rho}\sum_{\mu_1=3}^{2\rho n-1}\cdots\sum_{\substack{\mu_\nu=3\\ \mu_1+\ldots+\mu_{5\rho}=12\rho}}^{2m-1}\sum_{\mu_{\nu+1}=2}^{2}\cdots\sum_{\mu_{5\rho}=2}^{2} A_{\mu_1}\ldots A_{\mu_{5\rho}}\sum_{\nu=1}^{5m}\frac{1}{\nu!\,\mu_1!\ldots\mu_\nu!}$$

$$= 2^\sigma A_2^{-6\rho}\sum_{\mu_1=2}^{\infty}\cdots\sum_{\mu_{5\rho}=2}^{\infty} d'_{\mu_1,\ldots,\mu_{5\rho}}A_{\mu_1}\ldots A_{\mu_{5\rho}},$$

where $\mu_1+\ldots+\mu_{5\rho} = 12\rho$ and the d' are numerical constants. Thus the D_ρ have the required form (3).

Comparing (22) and (14) with (1) we see that in order to prove (1), we need now only show that

$$I_2, I_3 = O(A_2^{-\frac{1}{2}} n^{-(m-1)(s+1)^{-1}+\delta}). \tag{23}$$

For this purpose we estimate the modulus of the integrand of I_2 and I_3, which we shall denote by $G(\theta)$. Since

$$\left|\frac{1+e^{-\eta u+i\theta u}}{1+e^{-\eta u}}\right|^2 = 1-\frac{2e^{\eta u}(1-\cos\theta u)}{(1+e^{\eta u})^2},$$

we have, for any positive integer k,

$$\log G(\theta) = \sum_1^\infty{}_u \tfrac{1}{2}\log\{1-2(1-\cos\theta u)e^{\eta u}(e^{\eta u}+1)^{-2}\}$$

$$< -\tfrac{1}{2}\sum_1^k{}_u\, 2(1-\cos\theta u)e^{\eta u}(e^{\eta u}+1)^{-2}.$$

We write $\theta = 2\pi\alpha$, and choose k to satisfy

$$u_{k-1} < \eta^{-1} \leqslant u_k.$$

Then
$$\log G(\theta) < -c\sum_1^k{}_u\,(1-\cos 2\pi u\alpha)$$

$$< -c\sum_1^k{}_u\,\|u\alpha\|^2.$$

Thus, if $|\alpha| > \frac{1}{2}u_k^{-1}$, then by condition (II) and (5)

$$\log G(\theta) < -(m+1)(s+1)\log k, \qquad G(\theta) = O(n^{-m}). \tag{24}$$

On the other hand, if

$$\theta_0/2\pi \leqslant |\alpha| \leqslant \tfrac{1}{2}u_k^{-1},$$

then, by (16) and (17),

$$\sum_1^k{}_u\,\|u\alpha\|^2 = \alpha^2\sum_{\nu=1}^k u_k^2 \geqslant c\theta_0^2\sum_{\nu\leqslant\eta^{-1}}\nu^{2s(1-\delta)} \geqslant B^{-\frac{1}{2}}\eta^2.\eta^{-2s-1+3s\delta}$$

$$\geqslant B^{-\frac{1}{2}}\eta^{-1+\delta'} \geqslant n^{(-\frac{1}{2}+s)(s+1)^{-1}-\delta'} \geqslant n^e,$$

and (24) is again satisfied. Thus

$$I_2, I_3 = O(n^{-m}) = O(A_2^{-\frac{1}{2}} n^{-(m-1)+\delta}),$$

so that (23) is satisfied, and hence (1) is proved.

4. We shall show in this section that the sequences (i)–(iii) mentioned in § 1 (and many others) have the property (II). First we shall obtain a sufficient condition for the validity of (II). We suppose throughout that k is large.

Corresponding to each α in the range

$$\tfrac{1}{2}u_k^{-1} < \alpha \leqslant \tfrac{1}{2}, \tag{25}$$

there exist integers h, q with $(h,q) = 1$ such that

$$\left|\alpha - \frac{h}{q}\right| \leqslant (2u_k q)^{-1} \tag{26}$$

and

$$1 < q \leqslant 2u_k. \tag{27}$$

Here the possibility that $q = 1$ is excluded by (25) and (26). Now, if $\nu \leqslant k$ and $hu_\nu \equiv v \pmod q$ where $|v| \leqslant \frac{1}{2}q$, then by (26) we have

$$\left|u_\nu \alpha - \frac{hu_\nu}{q}\right| \leqslant (2q)^{-1}$$

and hence

$$\|u_\nu \alpha\| \geqslant \tfrac{1}{2}\|hu_\nu q^{-1}\| = \tfrac{1}{2}vq^{-1}.$$

Therefore in order to prove (II) it suffices to show that

$$S(q,h) = \sum_{v=-[\frac{1}{2}q]}^{-[\frac{1}{2}q]+q-1} v^2 N(q, h'v) > cq^2 \log^2 k, \tag{28}$$

where

$$hh' \equiv 1 \pmod q \tag{29}$$

and

$$N(q,w) = \sum_{\substack{\nu=1 \\ u_\nu \equiv w \pmod q}}^{k} 1. \tag{30}$$

We shall now consider each of the cases (i), (ii), (iii) of § 1 separately.

Case (i). We shall show quite generally that condition (28) is satisfied by every sequence u_k for which

(*a*) $s < \frac{3}{2}$,

and

(*b*) there exist constants k_0 and c such that, for all integers k, q satisfying $k > k_0$ and $1 < q \leqslant 18u_k/k$, at least $cq^2 \log^2 k$ of the numbers $u_1, u_2, \ldots, u_k$ are not divisible by q.

Clearly the sequence of primes, and in fact any sequence with $s < \frac{3}{2}$ and containing at least $ck^{2(s-1+\delta)}$ primes (for some positive δ) among its first k members, has this property.

Proof. If $q \leqslant 18u_k/k$, then the hypothesis (b) implies (28). If on the other hand $q > 18u_k/k$, then, since

$$\sum_{w=1}^{q} N(q,w) = k \quad \text{and} \quad N(q,w) \leqslant q^{-1}u_k + 1 \leqslant 3q^{-1}u_k$$

for each w by (27), we have

$$S(q,h) \geqslant 3q^{-1}u_k \sum_{v=1}^{[\frac{1}{2}k/(3q^{-1}u_k)]-1} v^2 \geqslant cq^2k^3u_k^{-2} > cq^2\log^2 k$$

since $[\frac{1}{2}k/(3q^{-1}u_k)] > 1$ and $s < \frac{3}{2}$. This settles case (i).

Case (ii). This is slightly more difficult than the previous one and appears to require exponential-sum methods in one form or another. We use such a method only implicitly, however, by referring to a lemma of Hua's.

Let $f(x)$ be a polynomial of degree s (> 1) (the case $s = 1$ is already settled) which takes integer values at integer places and let

$$N_1(k;q,w) = \sum_{\substack{\nu=1 \\ f(\nu)\equiv w \pmod q}}^{k} 1,$$

$$r_L(k;n) = \sum_{\substack{\nu_1=1 \\ f(\nu_1)+\ldots+f(\nu_L)=n}}^{k} \ldots \sum_{\nu_L=1}^{k} 1, \qquad R_L(k;q,h) = \sum_{\substack{n=1 \\ n\equiv h \pmod q}}^{\infty} r_L(k;n).$$

We note that $r_L(k;n) = 0$ if $n > Lf(k)$, so that the series defining $R_L(k;q,h)$ contains at most $Lf(k)q^{-1}+1$ non-zero terms.

LEMMA 2. (Main lemma in Hua.†) *If $L \geqslant 2^{s-1}$, then, for any $\delta > 0$,*

$$\sum_{n=1}^{\infty} r_L^2(k;n) \leqslant ck^{2L-s+\delta}.$$

LEMMA 3. *If L, q, H are integers satisfying*

$$L < c, \qquad 1 < q < ck^s, \qquad 1 \leqslant H < q/(2L), \qquad (h',q) = 1,$$

then

$$\Big|\sum_{v=-H}^{H} N_1(k;q,h'v)\Big|^{2L} \leqslant cq^{-1}Hk^s \sum_{n=1}^{\infty} r_L^2(k;n).$$

† *Quart. J. of Math.* (Oxford) 9 (1938), 199–202.

Proof. Since $2LH < q$ and, for each term of the multiple sum below, $|v_1+\ldots+v_L| \leqslant LH$, we have

$$\Big\{\sum_{v=-H}^{H} N_1(k;q,h'v)\Big\}^L = \sum_{v_1=-H}^{H}\ldots\sum_{v_L=-H}^{H} \sum_{\substack{\nu_1=1\\ f(\nu_1)\equiv h'v_1 (\mathrm{mod}\, q)}}^{k} \ldots \sum_{\substack{\nu_L=1\\ f(\nu_L)\equiv h'v_L (\mathrm{mod}\, q)}}^{k} 1$$

$$\leqslant \sum_{w=-LH}^{LH} \sum_{\substack{\nu_1=1\\ f(\nu_1)+\ldots+f(\nu_L)\equiv w (\mathrm{mod}\, q)}}^{k}\ldots\sum_{\nu_L=1}^{k} 1$$

$$= \sum_{w=-LH}^{LH} R_L(k;q,h'w).$$

Hence, using Cauchy's inequality, recalling the remark following the definition of R_L, and noting that $2LH < q$, we obtain

$$\Big\{\sum_{v=-H}^{H} N_1(k;q,h'v)\Big\}^{2L} \leqslant \Big\{\sum_{w=-LH}^{LH} R_L(k;q,h'w)\Big\}^2$$

$$\leqslant cH \sum_{w=-LH}^{LH} R_L^2(k;q,h'w)$$

$$\leqslant cH\{f(k)q^{-1}+1\}\sum_{w=-LH}^{LH}\sum_{\substack{n=1\\ n\equiv h'w (\mathrm{mod}\, q)}}^{\infty} r_L^2(k;n)$$

$$\leqslant cHq^{-1}k^s \sum_{n=1}^{\infty} r_L^2(k;n).$$

We now assume [see § 1 case (ii)] that corresponding to each integer $q > 1$ there exists an x_0 such that $f(x_0) \equiv w_0 \pmod q$, where $(w_0, q) = 1$. Now suppose that $1 < q \leqslant k^{\frac{1}{2}}$. Then

$$N(q, w_0) = N_1(k;q,w_0) > [kq^{-1}]-1 > cq^2\log^2 k,$$

which clearly implies (28). Suppose, on the other hand, that

$$k^{\frac{1}{2}} < q \leqslant 2f(k) < ck^s.$$

Then, writing $\qquad L = 2^{s-1}, \qquad H = [q^{1-1/(4s)}(2L)^{-1}]$

we have, by Lemmas 2 and 3,

$$\sum_{v=-H}^{H} N_1(k;q,h'v) < ck^{1+\delta}(H/q)^{1/(2L)} < \tfrac{1}{2}k,$$

for any integer h' satisfying $(h', q) = 1$. Hence, since

$$\sum_{v=-[\frac{1}{2}q]}^{-[\frac{1}{2}q]+q-1} N_1(k;q,h'v) = k,$$

we have, writing $N_1(k;q,h'v) = N(q,h'v)$,

$$S(q,h) > H^2 \sum_{\substack{v=-[\frac{1}{2}q] \\ |v|>H}}^{-[\frac{1}{2}q]+q-1} N(q,h'v) > \tfrac{1}{2}kH^2 > cq^2(kq^{-1/(2s)}) > cq^2k^{\frac{1}{2}},$$

which again implies (28). This completes the proof of case (ii).

Case (iii). This, of course, is the deepest of the three cases and requires, besides a refinement of Lemma 2, an estimate for the number of primes in an arithmetic progression. We shall retain the notations of case (ii).

LEMMA 4. *If* $L \geqslant 2^s+2$, *then*

$$\sum_{n=1}^{\infty} r_L^2(k;n) \leqslant ck^{2L-s}.$$

This lemma follows at once from the asymptotic formula for $r_L(k;n)$ given by Hua (loc. cit.). It could also be deduced by standard methods from Lemma 2.

LEMMA 5. *Let* $\pi(X,q,h)$ *denote the number of primes* $p \leqslant X$ *satisfying* $p \equiv h \pmod q$, *where* q *and* h *are coprime. Then, for any* $\delta > 0$, *there is a positive* $c = c(\delta)$ *such that*

$$\pi(X;q,h) = \frac{1}{\phi(q)} \int_2^X \frac{dt}{\log t} + O(Xe^{-A(\log X)^{\frac{1}{2}}}) + O\left(\frac{X^{1-cq^{-\delta}}}{\phi(q)\log X}\right),$$

where A *is a certain absolute constant. The constants implied by the O-symbol are independent of* q *and* h.

The estimate is obtained from the work of Siegel† on the magnitude of the class number by substituting Siegel's result $\sigma_1 \leqslant 1-ck^{-\epsilon}$ in the theorem given by Page.‡

Proof of Case (iii). We may assume $f(x)$ a polynomial of degree $s > 1$ taking only integral values at integral places, and that, corresponding to each integer $q > 1$, there exists an x_0 with $(x_0,q) = 1$ such that

$$f(x_0) \equiv w_0 \pmod q,$$

where $(w_0,q) = 1$. Suppose first that

$$1 < q \leqslant (\log k)^{5L_1}, \qquad L_1 = 2^s+2.$$

† *Acta Arithmetica* 1 (1935), 83–6. For a simpler proof see Estermann, *J. of London Math. Soc.* 23 (1948), 275–9.

‡ *Proc. London Math. Soc.* (2), 39 (1935), 116–41.

Writing $$N_2(k;q,w_0) = \sum_{\substack{\nu=1 \\ f(p_\nu)\equiv w_0 \pmod q}}^{k} 1,$$

we see from Lemma 5 that

$$N(q,w_0) = N_2(k;q,w_0) \geqslant \sum_{\substack{\nu=1 \\ p_\nu \equiv x_0 \pmod q}}^{k} 1 > c\frac{k}{\phi(q)} > cq^2\log^2 k,$$

so that (28) is satisfied.

Suppose, on the other hand, that

$$(\log k)^{5L_1} < q \leqslant 2f(p_k) < c(k\log k)^s.$$

Then, writing $H = [q/(\log k)^{4L_1}]$ and using Lemmas 3 and 4 with k replaced by p_k and L replaced by L_1, we obtain

$$\sum_{v=-H}^{H} N_2(k;q,h'v) \leqslant \sum_{v=-H}^{H} N_1(p_k;q,h'v) < cp_k(H/q)^{1/(2L_1)}$$

$$< ck\log k(\log k)^{-2} < \tfrac{1}{2}k.$$

But $$\sum_{v=-[\frac{1}{2}q]}^{-[\frac{1}{2}q]+q-1} N_2(k;q,h'v) = k,$$

and hence

$$S(q,h) > H^2 \sum_{\substack{v=-[\frac{1}{2}q]+q-1 \\ |v|>H}}^{-[\frac{1}{2}q]+q-1} N_2(k;q,h'v) > \tfrac{1}{2}kH^2 > cq^2\log^2 k.$$

5. By a slight modification in the argument used in the proof of formula (1), one can obtain an asymptotic expansion for $p_u^*(n)$, i.e. the number of solutions of

$$n = u_{\nu_1}+\ldots+u_{\nu_r} \quad (1 \leqslant \nu_1 \leqslant \ldots \leqslant \nu_r;\ r \text{ arbitrary}).$$

Using the generating function

$$F^*(w) = \prod_1^\infty{}_u\,(1-w^u)^{-1} = \sum_{n=0}^{\infty} p_u^*(n)w^n,$$

one obtains the asymptotic formula

$$p_u^*(n) = (2\pi A_2^*)^{-\frac{1}{2}}\exp\Big\{\sum_1^\infty{}_u\,(\eta^* u(e^{\eta^* u}-1)^{-1}-\log(1-e^{-\eta^* u}))\Big\}\times$$

$$\times\Big[1+\sum_{\rho=1}^{m-2} D_\rho^* + O(n^{-(m-1)(s+1)^{-1}+\delta})\Big], \quad (1^*)$$

where η^* is now determined from

$$n = \sum_1^\infty{}_u \, u(e^{\eta^* u}-1)^{-1},$$

and A_μ^* ($\mu = 2, 3, \ldots$) is defined by

$$A_\mu^* = \sum_1^\infty{}_u \, u^\mu g_\mu^*(e^{\eta^* u})(e^{\eta^* u}-1)^{-\mu}$$

for certain polynomials $g_\mu^*(x)$ of degree less than μ. Here $g_2^*(x) = x$ and D_ρ^* is constructed from the A_μ^* similarly to the D_ρ in (3). The relations (5) are also valid for the primed quantities. For the validity of (1*) condition (II) can be relaxed to

$$J_k^* = \inf_{\frac{1}{k}u_k^{-1}<\alpha\leqslant\frac{1}{2}} \left\{(\log k)^{-1} \sum_{\nu=1}^{k} \left(\frac{u_k}{u_\nu}\right)^2 \|u_\nu \alpha\|^2\right\} \to \infty \quad \text{when } k \to \infty. \quad \text{(II*)}$$

Finally, we note that in the particular case when

$$u_k = f(k) = ak^s + bk^{s-1} + \ldots,$$

formula (1*) gives

$$p_u^*(n) = \{1+O(n^{-(s+1)^{-1}})\}\left(\frac{as}{2\pi(s+1)}\right)^{\frac{1}{2}} \gamma^* n^{-\frac{3}{2}} \left(\frac{n}{\gamma^* a}\right)^{(1-b/a)(s+1)^{-1}} \times$$

$$\times \exp\left\{(s+1)\left(\frac{\gamma^{*s} n}{a}\right)^{(s+1)^{-1}} + c_f\right\},$$

where $$\gamma^* = s^{-1}\Gamma(1+s^{-1})\zeta(1+s^{-1})$$

and $$c_f = \prod_{\nu=1}^{s} (1+\alpha_\nu)(2\pi)^{-\frac{1}{2}s}, \quad \text{where } f(x) \equiv a \prod_{\nu=1}^{s} (x-\alpha_\nu).$$

ON THE HOMOTOPY GROUPS OF CERTAIN PAIRS AND TRIADS

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1. Introduction

An important problem is to determine the homotopy groups of complexes. If an induction on the dimensionality is attempted, an essential preliminary is to investigate the homotopy groups of the n-section of a complex relative to the $(n-1)$-section. It is these relative groups which are our present interest, and I supplement the results of Blakers and Massey (**4**). We use their methods of triads.

Let $(X; A, B)$ be a triad where X, A, B, and $C = A \cap B$ are pathwise-connected. Let (A, C) and (X, B) be $(n-1)$-connected, where $n \geqslant 3$. We suppose that $\pi_n(A, C)$ is free abelian, and is isomorphic to $\pi_n(X, B)$ under the injection. If $r \geqslant n+2$, let

$$\mathrm{P}\colon \pi_{r-n+1}(B, C) \otimes \pi_n(A, C) \to \pi_r(X; A, B)$$

be the homomorphism whose value on the tensor product $\alpha \otimes \beta$ of $\alpha \in \pi_{r-n+1}(B, C)$ and $\beta \in \pi_n(A, C)$ is the triad Whitehead product $[\alpha, \beta]$, as defined in (**3**) or (**8**). Let (B, C) be l-connected and let C be m-connected ($m \geqslant 1$). With these assumptions I shall prove the following two theorems† in § 3.

THEOREM (1.1). *Let* $p = l+n-1+\min(l, m)$. *Suppose that*‡

$$H_r(A, C) = 0$$

if $n+1 \leqslant r \leqslant p$ *and that* $H_r(X, B) = 0$ *if* $n+1 \leqslant r \leqslant p+1$. *Then, if* $n+2 \leqslant r \leqslant p$, *the homomorphism*

$$\mathrm{P}\colon \pi_{r-n+1}(B, C) \otimes \pi_n(A, C) \to \pi_r(X; A, B)$$

has a left inverse.

THEOREM (1.2). *Let* $q = l+n-1+\min(l, m, n-1)$. *Suppose that* $H_r(A, C) = 0$ *if* $n+1 \leqslant r \leqslant q$ *and that* $H_r(X, B) = 0$ *if* $n+1 \leqslant r \leqslant q+1$. *Then, if* $n+2 \leqslant r \leqslant q$,

$$\mathrm{P}\colon \pi_{r-n+1}(B, C) \otimes \pi_n(A, C) \approx \pi_r(X; A, B).$$

† I am glad to acknowledge the help of Dr. J. C. Moore in connexion with these theorems.

‡ Here and in (1.2), H refers to the singular homology theory with integer coefficients.

We apply (1.2) in the upper homotopy sequence of the triad,

$$\ldots \to \pi_r(A, C) \xrightarrow{i} \pi_r(X, B) \xrightarrow{j} \pi_r(X; A, B) \xrightarrow{\delta} \pi_{r-1}(A, C) \to \ldots,$$

in which i and j are injections and δ is the boundary operator. Hence under the conditions of (1.2) we have

COROLLARY (1.3). *The following sequence is exact*:

$$\pi_q(A, C) \xrightarrow{i} \ldots$$

$$\ldots \to \pi_r(A, C) \xrightarrow{i} \pi_r(X, B) \xrightarrow{\mathrm{P}^{-1}j} \pi_{r-n+1}(B, C) \otimes \pi_n(A, C) \xrightarrow{\delta\mathrm{P}} \to \pi_{r-1}(A, C) \to \ldots.$$

In (1.3), $\delta\mathrm{P}$ may be replaced by $Q(\delta' \otimes 1)$,

$$\pi_{r-n+1}(B, C) \otimes \pi_n(A, C) \xrightarrow{\delta' \otimes 1} \pi_{r-n}(C) \otimes \pi_n(A, C) \xrightarrow{Q} \pi_{r-1}(A, C),$$

where $\delta' \otimes 1$ is the tensor product of the boundary operator and the identity, and $Q(\alpha \otimes \beta)$ is the relative Whitehead product $[\alpha, \beta]$. For by (4.3) of (**3**) or (2.18) of (**8**) we have

$$\delta\mathrm{P} = \pm Q(\delta' \otimes 1).$$

If B is contractible, as it is in the following application, then $\delta' \otimes 1$ is an isomorphism.

If (X, A) is a pair of spaces, we represent elements of $\pi_n(X, A)$ by maps

$$(V^n, S^{n-1}) \to (X, A),$$

where V^n is an n-dimensional element and S^{n-1} is its boundary sphere. Hence, if $n \geqslant 3$ and $r < 2n-2$, the composition homomorphism

$$\pi_n(X, A) \otimes \pi_r(V^n, S^{n-1}) \to \pi_r(X, A)$$

is defined. Let K be an m-connected CW-complex ($m \geqslant 1$) and let K^n denote the n-section of K. Let M be a cone on K^{n-1} as base which is disjoint from $K^n - K^{n-1}$, and let $L = K^n \cup M$. Then, if $n \geqslant 3$ and $r < 2n-2$, we have the following naturality diagram, in which i and i' are injections, and ψ and ψ' are the composition homomorphisms,

$$\begin{array}{ccc} \pi_n(K^n, K^{n-1}) \otimes \pi_r(V^n, S^{n-1}) & \xrightarrow{\psi} & \pi_r(K^n, K^{n-1}) \\ \downarrow{\scriptstyle i' \otimes 1} & & \downarrow{\scriptstyle i} \\ \pi_n(L, M) \otimes \pi_r(V^n, S^{n-1}) & \xrightarrow[\psi']{} & \pi_r(L, M). \end{array}$$

Now i' is an isomorphism, and it follows from (23.1) of (**6**) that ψ' is an isomorphism. Hence i has a right inverse which embeds $\pi_r(L, M)$ in $\pi_r(K^n, K^{n-1})$ as

$$\psi\{\pi_n(K^n, K^{n-1}) \otimes \pi_r(V^n, S^{n-1})\}.$$

In (1.3) we take $(X; A, B) = (L; K^n, M)$ and substitute $Q(\delta' \otimes 1)$ for δP. Then, since $\delta' \otimes 1$ is an isomorphism, we deduce

THEOREM (1.4). *Let K be m-connected ($m \geqslant 1$), and let $n \geqslant 3$. Then, if $r < \min(2m+n, 2n-2)$, we have the injective direct sum*†

$$\psi+Q\colon \{\pi_n(K^n, K^{n-1}) \otimes \pi_r(V^n, S^{n-1})\}+\{\pi_{r-n+1}(K^{n-1}) \otimes \pi_n(K^n, K^{n-1})\} \approx \pi_r(K^n, K^{n-1}).$$

If $p \geqslant 1$ and $p+q \geqslant 3$, the group $\mathscr{H}_{p,q}(K)$ in the Massey homology spectrum [see (**6**)] is defined as the kernel of ∂_1, factored by the image of ∂_0, where ∂_0 and ∂_1 are the boundary homomorphisms

$$\pi_{p+q+1}(K^{p+1}, K^p) \xrightarrow{\partial_0} \pi_{p+q}(K^p, K^{p-1}) \xrightarrow{\partial_1} \pi_{p+q-1}(K^{p-1}, K^{p-2}).$$

Let G_q denote the stable group of the q-stem in the homotopy groups of spheres. Then arguing as in §§ 12–16 of (**6**) we deduce from (1.4)

THEOREM (1.5). *Let K be m-connected ($m \geqslant 1$). Then, if*

$$0 \leqslant q < \min(p-3, 2m),$$

we have the direct sum

$$\mathscr{H}_{p,q}(K) \approx H_p(K, G_q)+H_p(K, \pi_{q+1}(K)).$$

2. The groups $\pi_r(Y^*, Y)$

Let Y be a 1-connected CW-complex and let $Y^* = Y \cup e^n$ be the complex formed by attaching an n-cell. This section is devoted to a study of the groups $\pi_r(Y^*, Y)$. We apply (1.3) in two different ways, and in each case obtain a useful exact sequence which generalizes the sequence in Theorem 1 of (**10**). We proceed at once to the first of these. Let $n \geqslant 3$, let $r \geqslant n+2$, and let $r \leqslant 3n-4$. Let E^n be the n-fold suspension isomorphism, let H be the generalized Hopf invariant of (**9**), and let α_* be induced by the attaching map $\alpha\colon S^{n-1} \to Y$, as follows:

$$\pi_r(S^n) \xrightarrow{H} \pi_r(S^{2n-1}) \xleftarrow{E^n} \pi_{r-n}(S^{n-1}) \xrightarrow{\alpha_*} \pi_{r-n}(Y).$$

We write $H_\alpha = \alpha_* E^{-n} H$. Let χ be induced by shrinking Y to a point, and let $Q(\gamma)$ ($\gamma \in \pi_{r-n}(Y)$) be the relative Whitehead product of γ and a generator of $\pi_n(Y^*, Y)$, as shown:

$$\chi\colon \pi_r(Y^*, Y) \to \pi_r(S^n), \qquad Q\colon \pi_{r-n}(Y) \to \pi_{r-1}(Y^*, Y).$$

† $\psi+Q$ is the homomorphism whose value on the pair (α, β) is $\psi\alpha+Q\beta$.

Assuming (1.2), I shall prove

THEOREM (2.1). *Let Y be m-connected and let $p = \min(3n-4, 2m+n)$. Then the following sequence is exact:*

$$\pi_p(Y^*, Y) \xrightarrow{\chi} \cdots$$

$$\cdots \to \pi_r(Y^*, Y) \xrightarrow{\chi} \pi_r(S^n) \xrightarrow{H_\alpha} \pi_{r-n}(Y) \xrightarrow{Q} \pi_{r-1}(Y^*, Y) \to \cdots.$$

In case $(Y^*, Y) = (V^n, S^{n-1})$ and $\alpha = 1$, (2.1) is essentially the same as Theorem 1 of (**10**). For let $\pi_r(V^n, S^{n-1})$ be identified with $\pi_{r-1}(S^{n-1})$ by the boundary isomorphism. Then it follows from (2.1) that the sequence

$$\pi_{3n-5}(S^{n-1}) \xrightarrow{E} \cdots$$

$$\cdots \to \pi_{r-1}(S^{n-1}) \xrightarrow{E} \pi_r(S^n) \xrightarrow{H'} \pi_{r-n}(S^{n-1}) \xrightarrow{Q'} \pi_{r-2}(S^{n-1}) \to \cdots \tag{2.2}$$

is exact, where $H' = E^{-n}H$ and $Q'(\gamma)$ is the ordinary Whitehead product of $\gamma \in \pi_{r-n}(S^{n-1})$ and a generator of $\pi_{n-1}(S^{n-1})$. Using (3.59) of (**9**), the sequence of (**10**) is easily transformed into (2.2).

The proof of (2.1) is as follows. Let Z be a cone on Y as base which is disjoint from e^n, and let $Z^* = Z \cup Y^*$. Since $\pi_n(Y^*, Y)$ is cyclic infinite, we identify

$$G \otimes \pi_n(Y^*, Y) = G,$$

where G is any abelian group. We apply (1.2) to the triad $(Z^*; Y^*, Z)$, for which

$$q = \min(2m+n, m+2n-1)$$

since Y is m-connected and $l = m+1$. Hence, if $r \leqslant q$ in the following diagram, $\mathrm{P}^{-1}j$ is a homomorphism.

$$\begin{array}{ccccccc}
 & & \pi_r(Z^*, Z) & \xrightarrow{\mathrm{P}^{-1}j} & \pi_{r-n+1}(Z, Y) & & \\
 & \overset{i}{\nearrow} & \downarrow{\scriptstyle \chi'} & & \downarrow{\scriptstyle S'} & \searrow^{\delta \mathrm{P}} & \\
\pi_r(Y^*, Y) & & & & & & \pi_{r-1}(Y^*, Y). \\
 & \underset{\chi}{\searrow} & \pi_r(S^n) & \xrightarrow[h]{} & \pi_{r-n}(Y) & \underset{Q}{\nearrow} &
\end{array} \tag{2.3}$$

In (2.3), χ' is induced by shrinking Z to a point, so that $\chi = \chi' i$, and χ' is an isomorphism since Z is contractible. For the same reason the boundary operator δ' is an isomorphism. Hence it follows from (1.3) that the sequence χ, h, Q,..., is exact starting with χ: $\pi_q(Y^*, Y) \to \pi_q(S^n)$, where $h = \delta' \mathrm{P}^{-1} j (\chi')^{-1}$. Since $q \geqslant p = \min(3n-4, 2m+n)$, this proves (2.1) apart from the identification:

$$h = \pm H_\alpha \colon \pi_r(S^n) \to \pi_{r-n}(Y) \quad \text{if } r \leqslant p. \tag{2.4}$$

We proceed to prove (2.4). Let E^n_+ and E^n_- be the hemispheres of S^n, so that $E^n_+ \cup E^n_- = S^n$ and $E^n_+ \cap E^n_- = S^{n-1}$. Let

$$\theta\colon (S^n; E^n_+, E^n_-) \to (Z^*; Y^*, Z)$$

be a map which agrees with α on S^{n-1} and is such that $\theta \mid (E^n_+ - S^{n-1})$ is a homeomorphism onto e^n. Then, if θ_* denotes the homomorphisms induced by θ, we have a naturality diagram, as follows, in which j_0, P_0, and δ'_0 refer to the triad $(S^n; E^n_+, E^n_-)$ and we place no limit on r.

$$\begin{array}{ccccccc}
\pi_r(S^n, E^n_-) & \xrightarrow{j_0} & \pi_r(S^n; E^n_+, E^n_-) & \xleftarrow{\mathrm{P}_0} & \pi_{r-n+1}(E^n_-, S^{n-1}) & \xrightarrow{\delta'_0} & \pi_{r-n}(S^{n-1}) \\
\downarrow{\scriptstyle \theta_*} & & \downarrow{\scriptstyle \theta_*} & & \downarrow{\scriptstyle \theta_*} & & \downarrow{\scriptstyle \alpha_*} \\
\pi_r(Z^*, Z) & \xrightarrow[j]{} & \pi_r(Z^*; Y^*, Z) & \xleftarrow[\mathrm{P}]{} & \pi_{r-n+1}(Z, Y) & \xrightarrow[\delta']{} & \pi_{r-n}(Y).
\end{array}$$

If $r \leqslant 3n-4$, P_0 is an isomorphism, by (1.2), and then

$$h_0 = \delta'_0 \mathrm{P}_0^{-1} j_0 (\chi'_0)^{-1}\colon \pi_r(S^n) \to \pi_{r-n}(S^{n-1})$$

is the homomorphism h for the triad $(S^n; E^n_+, E^n_-)$, where

$$\chi'_0 = \chi' \theta_*\colon \pi_r(S^n, E^n_-) \to \pi_r(S^n)$$

is induced by shrinking E^n_- to a point. If $r \leqslant p$, both P and P_0 are isomorphisms, both h and h_0 are defined, and so, by reference to the diagram,

$$\alpha_* h_0 \chi'_0 = \alpha_* \delta'_0 \mathrm{P}_0^{-1} j_0 = \delta' \mathrm{P}^{-1} j \theta_* = h \chi' \theta_*.$$

Therefore $\alpha_* h_0 = h$, and (2.4) follows from

LEMMA (2.5). *$E^n h_0 = \pm H\colon \pi_r(S^n) \to \pi_r(S^{2n-1})$ if $r \leqslant 3n-4$.*

To prove (2.5) we study the following diagram:

$$\begin{array}{ccccc}
\pi_{r-n}(S^{n-1}) & \xleftarrow{h_0} & \pi_r(S^n) & \xrightarrow{h_0} & \pi_{r-n}(S^{n-1}) \\
\uparrow{\scriptstyle \delta'_0} & & \downarrow{\scriptstyle I} & & \downarrow{\scriptstyle E^{n+1}} \\
\pi_{r-n+1}(E^n_-, S^{n-1}) & \xrightarrow[\mathrm{P}_0]{} & \pi_r(S^n; E^n_+, E^n_-) & \xrightarrow[\bar{H}]{} & \pi_{r+1}(S^{2n}).
\end{array} \tag{2.6}$$

In (2.6), I is the injection, so that

$$\pm I = j_0 (\chi'_0)^{-1} = \mathrm{P}_0 (\delta'_0)^{-1} h_0,$$

by the definition of h_0. Also $\bar{H}$ is the homomorphism defined in (3.2) of (**8**), but there denoted by H, such that $\bar{H} I = \pm EH$. It follows from (2.24) and (3.5) of (**8**) that $\bar{H} \mathrm{P}_0 = \pm E^{n+1} \delta'_0$, and so

$$EH = \pm \bar{H} I = \pm \bar{H} \mathrm{P}_0 (\delta'_0)^{-1} h_0 = \pm E^{n+1} h_0.$$

Since $E\colon \pi_r(S^{2n-1}) \approx \pi_{r+1}(S^{2n})$, this proves (2.5) and completes the proof of (2.1).

We notice the following corollary.† Suppose that $E\colon \pi_{r-1}(S^{n-1}) \approx \pi_r(S^n)$ for a certain value of r. Then the composition

$$\pi_r(E_+^n, S^{n-1}) \xrightarrow{\theta_*} \pi_r(Y^*, Y) \xrightarrow{\chi} \pi_r(S^n)$$

is an isomorphism [(**5**) § 1]. Hence χ has a right inverse, and from (2.1) we obtain the direct-sum decomposition

$$\pi_r(Y^*, Y) = Q\pi_{r-n+1}(Y) + \theta_* \pi_r(E_+^n, S^{n-1}) \tag{2.7}$$

if $r < \min(3n-4, 2m+n)$. Here θ_* is an isomorphism into, but the kernel of Q is $H_\alpha \pi_{r+1}(S^n)$.

The foregoing is one approach to the problem of this section. For the other approach we apply (1.3) in another triad. Let M be the mapping cylinder of $\alpha\colon S^{n-1} \to Y$, so that $M \cap E_+^n = S^{n-1}$. We identify

$$Y^* = E_+^n \cup M,$$

so that the pair (Y^*, Y) is a deformation retract of the pair (Y^*, M). Let Y^* be l-connected. Then, if $r \leqslant l+2n-3$, it follows from Theorem I of (**2**), applied to the triad $(Y^*; E_+^n, M)$, that the injection

$$\pi_{r-n+1}(M, S^{n-1}) \to \pi_{r-n+1}(Y^*, E_+^n)$$

is an isomorphism. We factor this into an isomorphism

$$\xi\colon \pi_{r-n+1}(M, S^{n-1}) \to \pi_{r-n+1}(Y^*)$$

followed by injection into $\pi_{r-n+1}(Y^*, E_+^n)$. In (1.3), applied with

$$(X; A, B) = (Y^*; E_+^n, M),$$

we identify $\pi_{r-n+1}(M, S^{n-1})$ with $\pi_{r-n+1}(Y^*)$ by ξ, $\pi_r(E_+^n, S^{n-1})$ with $\pi_{r-1}(S^{n-1})$ by the boundary isomorphism ∂, and $\pi_r(Y^*, M)$ with $\pi_r(Y^*, Y)$ by the inverse of the injection. The result is (2.8) below. As the reader will easily verify, the following are homomorphisms to replace those in (1.3) when we make the above identifications.

If $r \leqslant \min(2l+n-1, l+2n-3)$ let $H_Y = \xi \mathrm{P}^{-1} k$, where k is the injection:

$$\pi_r(Y^*, Y) \xrightarrow{k} \pi_r(Y^*; E_+^n, M) \xleftarrow{\mathrm{P}} \pi_{r-n+1}(M, S^{n-1}) \xrightarrow{\xi} \pi_{r-n+1}(Y^*).$$

If $r \leqslant 3n-4$ let $\mathrm{P}_Y = Q' E^{-1} \zeta$,

$$\pi_{r-n+1}(Y^*) \xrightarrow{\zeta} \pi_{r-n+1}(S^n) \xleftarrow{E} \pi_{r-n}(S^{n-1}) \xrightarrow{Q'} \pi_{r-2}(S^{n-1}),$$

† Some examples of this and other applications of (2.1) were earlier obtained quite otherwise by Dr. M. G. Barratt, and will shortly be published.

where Q' is the same as in (2.2) and ζ is induced by shrinking $Y \subset Y^*$ to a point. Finally, let $\phi = \theta_* \partial^{-1}$, where

$$\pi_{r-1}(S^{n-1}) \xleftarrow{\partial} \pi_r(E^n_+, S^{n-1}) \xrightarrow{\ell_*} \pi_r(Y^*, Y).$$

Then we have

THEOREM (2.8). *Let Y^* be l-connected and let $q = \min(3n-4, 2l+n-1)$. Then the following sequence is exact:*

$$\pi_{q-1}(S^{n-1}) \xrightarrow{\phi} \ldots$$
$$\ldots \to \pi_{r-1}(S^{n-1}) \xrightarrow{\phi} \pi_r(Y^*, Y) \xrightarrow{H_Y} \pi_{r-n+1}(Y^*) \xrightarrow{\mathrm{P}_Y} \pi_{r-2}(S^{n-1}) \to \ldots.$$

The analogue of (2.7) for this theorem is

COROLLARY (2.9). *Suppose that $E\colon \pi_{r-1}(S^{n-1}) \approx \pi_r(S^n)$. Then, if $r \leqslant q$, we have the direct-sum decomposition*

$$\eta\colon \pi_r(Y^*, Y) \approx \pi_r(S^n) + \mathrm{P}_Y^{-1}(0),$$

in which $\eta(\gamma) = \chi(\gamma) + H_Y(\gamma)$ $(\gamma \in \pi_r(Y^*, Y))$.

3. Proof of the main theorems

We prove (1.1) and (1.2) simultaneously. The idea of the proof is to obtain these results for complexes by shrinking $C = A \cap B$ to a point, the extension to general spaces following from an application of the realization of singular complexes given by J. H. C. Whitehead in (**12**). It is convenient to carry out the proof in four stages of increasing generality according to the class of triad considered.

Let K be an l-connected CW-complex ($l \geqslant 1$), and let K and S^n be embedded in $K \times S^n$ as the axes of the product, so that they meet in a vertex of K, say y. Let $\tilde{K}$ be the n-fold suspension of K. Then, as is shown in § 2 of (**8**), $\tilde{K}$ is obtained from $K \times S^n$ by shrinking $K \cup S^n$ to a point. By Theorem II of (**2**) the induced homomorphism

$$\theta\colon \pi_r(K \times S^n, K \cup S^n) \to \pi_r(\tilde{K})$$

is an isomorphism if $r \leqslant u = l+n+\min(l, n-1)$. Let $r \geqslant n+2$ and let

$$\mathrm{C}\colon \pi_{r-n}(K) \to \pi_r(K \times S^n, K \cup S^n)$$

be the homomorphism whose value on $\alpha \in \pi_{r-n}(K)$ is the cross, or star, product of α and a generator of $\pi_n(S^n)$. By (2.10) of (**8**) the n-fold suspension homomorphism is

$$\pm\theta\mathrm{C}\colon \pi_{r-n}(K) \to \pi_r(\tilde{K}).$$

Therefore, by Lemma 5 of (**2**), $\theta\mathrm{C}$ is an isomorphism if $r \leqslant 2l+n$.

Therefore C has a left inverse if $r \leqslant 2l+n$, and C is an isomorphism if $r \leqslant u$. By (8.3) of (**3**) [cf. § 3 of (**8**)] the boundary operator

$$\delta\colon \pi_r(K \times S^n, K \cup S^n) \to \pi_{r-1}(S^n \cup K; S^n, K)$$

is an isomorphism. Also it follows from (5.7) of (**3**) or (2.1) of (**8**) that $\delta C(\alpha)$ $(\alpha \in \pi_{r-n}(K))$ is the triad Whitehead product of α and a generator of $\pi_n(S^n)$. Hence, altering r to $r+1$ in conformity with § 1, we have

LEMMA (3.1). *The homomorphism*

$$\mathrm{P}\colon \pi_{r-n+1}(K) \otimes \pi_n(S^n) \to \pi_r(S^n \cup K; S^n, K)$$

has a left inverse if $r < 2l+n$, *and* P *is an isomorphism if*

$$r < u = l+n+\min(l, n-1).$$

Lemma (3.1) is the first stage in proving our two theorems. In the second stage we replace S^n by a cluster of spheres of dimension n, and prove the corresponding results. Consider a set of n-cells e_j^n which are disjoint from one another and for each of which $S_j^n = e_j^n \cup y$ is an n-sphere. Let Π denote the product complex of these n-spheres, with the weak topology, and let $\Sigma \subset \Pi$ denote the union of the spheres, embedded as the set of axes of the product in the usual way. Let K be as before, so that y is a vertex of K, and suppose that $\Pi \cap K = y$. It follows from (3.1) that each of the homomorphisms

$$\mathrm{P}\colon \pi_{r-n+1}(K) \otimes \pi_n(S_j^n) \to \pi_r(S_j^n \cup K; S_j^n, K)$$

has a left inverse if $r < 2l+n$ and is an isomorphism if $r < u$. Taking the direct sum on both sides we deduce that the homomorphism

$$\mathrm{P}\colon \pi_{r-n+1}(K) \otimes \pi_n(\Pi) \to \pi_r(\Pi \cup K; \Pi, K)$$

has a left inverse if $r < 2l+n$ and is an isomorphism if $r < u$. Now the injection

$$i\colon \pi_r(\Sigma \cup K; \Sigma, K) \to \pi_r(\Pi \cup K; \Pi, K)$$

is an isomorphism if $r \leqslant l+2n-2$. This follows by applying the five lemma [Lemma 7 of (**1**)] to the injection of the lower homotopy sequence of the triad $(\Sigma \cup K; \Sigma, K)$ into that of $(\Pi \cup K; \Pi, K)$ since by Lemma 5 of (**2**) the triad $(\Pi \cup K; \Sigma \cup K, \Pi)$ is $(l+2n-1)$-connected. Let

$$i'\colon \pi_n(\Sigma) \approx \pi_n(\Pi)$$

be the injection isomorphism. Then we have the naturality diagram:

$$\begin{array}{ccc} \pi_{r-n+1}(K) \otimes \pi_n(\Sigma) & \xrightarrow{\mathrm{P}} & \pi_r(\Sigma \cup K; \Sigma, K) \\ \downarrow{\scriptstyle 1\otimes i'} & & \downarrow{\scriptstyle i} \\ \pi_{r-n+1}(K) \otimes \pi_n(\Pi) & \xrightarrow[\mathrm{P}]{} & \pi_r(\Pi \cup K; \Pi, K). \end{array}$$

Taking these conclusions together we obtain

LEMMA (3.2). *The homomorphism*

$$\mathrm{P}\colon \pi_{r-n+1}(K) \otimes \pi_n(\Sigma) \to \pi_r(\Sigma \cup K; \Sigma, K)$$

has a left inverse if $r < 2l+n$, *and* P *is an isomorphism if* $r < u$.

In the third stage of the proof let L be a CW-complex and let M be a subcomplex. Let M^* be a CW-complex formed by adding n-cells ($n \geqslant 3$), disjoint from $L-M$, to M and let $L^* = L \cup M^*$. We prove

LEMMA (3.3). *Let* (L, M) *be* l*-connected* ($l \geqslant 1$) *and let* M *be* m*-connected* ($m \geqslant 1$). *Then the homomorphism*

$$\mathrm{P}\colon \pi_{r-n+1}(L, M) \otimes \pi_n(M^*, M) \to \pi_r(L^*; M^*, L)$$

has a left inverse if $r \leqslant p = l+n-1+\min(l, m)$, *and* P *is an isomorphism if* $r \leqslant q = l+n-1+\min(l, m, n-1)$.

Let $\bar{M}$ be a cone on M as base which is disjoint from L^*-M, and let $\bar{L}^* = L^* \cup \bar{M}$, $\bar{M}^* = M^* \cup \bar{M}$, and $\bar{L} = L \cup \bar{M}$. The complex formed from $\bar{L}^*$ by shrinking $\bar{M}$ to a point is of the form $\Sigma \cup K$ considered in (3.2), where the cells of M^*-M correspond to the spheres of Σ and K is formed from L by shrinking M to a point. Let η be the injection, and let θ be induced by the identification map:

$$\pi_r(L^*; M^*, L) \xrightarrow{\eta} \pi_r(\bar{L}^*; \bar{M}^*, L) \xrightarrow{\theta} \pi_r(\Sigma \cup K; \Sigma, K).$$

It follows from Theorem 12 of (**11**) that θ is an isomorphism. Also η, and hence $\theta\eta$, is an isomorphism if $r < l+m+n$ by Corollary (3.5) of (**7**). Also the homomorphism

$$\pi_r(L, M) \to \pi_r(K)$$

induced by the identification map is an isomorphism if $r \leqslant l+m$. In particular K is l-connected. Therefore in the following naturality diagram the vertical homomorphisms, which are those induced by the identification map, are both isomorphisms if $r < l+m+n$:

$$\begin{array}{ccc} \pi_{r-n+1}(L, M) \otimes \pi_n(M^*, M) & \xrightarrow{\mathrm{P}} & \pi_r(L^*; M^*, L) \\ \downarrow & & \downarrow \\ \pi_{r-n+1}(K) \otimes \pi_n(\Sigma) & \xrightarrow[\mathrm{P}]{} & \pi_r(\Sigma \cup K; \Sigma, K). \end{array}$$

Hence and from (3.2), (3.3) follows at once.

We now begin the last stage, in which (1.1) and (1.2) are proved. Let $(X; A, B)$ be as in § 1, so that (B, C) is l-connected and C is m-connected ($m \geqslant 1$). We assume that $l \geqslant 1$ since both theorems are vacuous if

$l = 0$. Let (L, M) be the singular complexes of the pair (B, C), realized as in Chap. V of (**12**), so that L is a CW-complex and M is a subcomplex. Then by Theorem 22 of (**12**) and the five lemma the natural map

$$k\colon (L, M) \to (B, C)$$

induces an isomorphism of the homotopy sequences of the pairs (L, M) and (B, C). Let ∂ be the boundary homomorphism and let k_* be induced by k:

$$\pi_n(A, C) \xrightarrow{\partial} \pi_{n-1}(C) \xleftarrow{k_*} \pi_{n-1}(M).$$

By hypothesis $\pi_n(A, C)$ is a free abelian group. For each element g of its basis we attach an n-cell to M by a cellular map representing

$$k_*^{-1}\partial(g) \in \pi_{n-1}(M).$$

Let M^* denote the resultant complex, which we assume is disjoint from $L - M$, and let $L^* = M^* \cup L$. Let

$$k'\colon (L^*; M^*, L) \to (X; A, B)$$

be a map which agrees with k on L, and which induces an isomorphism $\pi_n(M^*, M) \approx \pi_n(A, C)$. Throughout what follows it is to be understood that the homomorphisms, apart from **P**, are those induced by k'.

Suppose that $H_r(A, C) = 0$ if $n < r \leqslant s$ and that $H_r(X, B) = 0$ if $n < r \leqslant s+1$, where $s = p$ in the case of (1.1) and $s = q$ in the case of (1.2). Since $H_r(M^*, M) \approx H_r(A, C)$ if $r \leqslant s$ and since $H_r(M) \approx H_r(C)$ for all r, it follows from Theorem (3.4) of (**7**) that $\pi_r(M^*, M) \to \pi_r(A, C)$ is onto if $r = s$ and is an isomorphism if $r < s$. Similarly

$$\pi_r(L^*, L) \to \pi_r(X, B)$$

is onto if $r = s+1$ and is an isomorphism if $r \leqslant s$. Hence by the five lemma we have

$$\pi_r(L^*; M^*, L) \approx \pi_r(X; A, B)$$

if $r \leqslant s$. Hence (1.1) and (1.2) follow from (3.3) and the following naturality diagram:

$$\begin{array}{ccc} \pi_{r-n+1}(L, M) \otimes \pi_n(M^*, M) & \xrightarrow{\mathbf{P}} & \pi_r(L^*; M^*, L) \\ \downarrow & & \downarrow \\ \pi_{r-n+1}(B, C) \otimes \pi_n(A, C) & \xrightarrow{\mathbf{P}} & \pi_r(X; A, B). \end{array}$$

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HOMOTOPY RINGOIDS AND HOMOTOPY GROUPS

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1. Introduction

If X, Y are CW-complexes, the set of homotopy classes of maps $X \to Y$ may, under certain circumstances, be given a group structure. This gives rise to the homotopy groups, Borsuk's cohomotopy groups and their generalizations by Spanier and Whitehead, and to track groups. If C is a collection of suitable spaces, the multiplication by composition of homotopy classes of maps between the spaces of C may be distributive over a suitably defined addition; I consider such collections, and use the name *ringoid* for the algebraic system $\Pi(C)$ so constructed.

I show how to compute the ringoid associated with a collection of finite $(n-1)$-connected complexes of dimension less than $n+2$; this rests on a calculation of the structure of the ringoid of a particular collection Y^{n+1}, of spaces $S^n \cup e^{n+1}$.

I use these concepts in the latter part of the paper to compute the first $2n-2$ homotopy groups of spaces of the type $S^n \cup e^{n+1}$ in terms of the homotopy groups of spheres, and hence the structure of the ringoid of $C = (Y^{n+1}, S^m)$ $(m \leqslant 2n-2,\ n \leqslant 2m-3)$. In a later paper I shall discuss the $(2n-1)$th and $2n$th homotopy groups of these spaces.

I am deeply indebted to Professor J. H. C. Whitehead for his helpful advice and criticism.

2. On ringoids $\Pi(C_r^N)$

Let $(G_{i,j})$ be a collection of disjoint abelian groups, written additively, one for each ordered pair of symbols (i, j) in some indexing set. Suppose that there is a bilinear product

$$G_{i,j} \circ G_{j,k} \subset G_{i,k}$$

for every triple (i, j, k). I describe such a collection as a *ringoid* if every $G_{i,i}$ possesses an element 1_i, necessarily unique, such that, for every $g_{i,j} \in G_{i,j}$,

$$1_i \circ g_{i,j} = g_{i,j} = g_{i,j} \circ 1_j.$$

Let S be the reduced suspension functor as defined in § 3 of (**6**), which associates with every space X another SX, with every map $f\colon X \to Y$

another $Sf\colon SX \to SY$, and with every homotopy class ν of maps $X \to Y$, written

$$\nu\colon X \to Y,$$

a homotopy class $S_*\nu\colon SX \to SY$.

Let $r > 1$, let C_r^N be a collection of $(r-1)$-connected CW-complexes of dimension not exceeding N, and let $\Pi(C_r^N)$ be the collection of homotopy classes of maps between the spaces of C_r^N. We suppose that $N \leqslant 2r-2$, and turn $\Pi(C_r^N)$ into a ringoid.

Let SC_r^N be the collection of spaces SX for all $X \in C_r^N$. It follows from Theorem (5.1) in (**6**) that S_* maps $\Pi(C_r^N)$ on $\Pi(SC_r^N)$ in (1, 1) fashion, and hence $S_*^2 = S_* \circ S_*$ maps $\Pi(C_r^N)$ on $\Pi(S^2C_r^N)$ in (1, 1) fashion. The homotopy classes of maps $S^2X \to S^2Y$ form an additive abelian group under track addition [see (**1**, **6**)]; let X_i, $X_j \in C_r^N$, and let $G_{i,j}$ be the collection of homotopy classes of maps $X_j \to X_i$. Turn $G_{i,j}$ into a group by

$$(\alpha+\beta) = (S_*^2)^{-1}(S_*^2\alpha + S_*^2\beta) \quad (\alpha, \beta \in G_{i,j}). \tag{2.1}$$

The identity element of $G_{i,j}$, written† $0_{i,j}$, is the homotopy class of all the maps of X_j into some point of X_i.

If $\gamma \in G_{j,k}$, let $\alpha \circ \gamma$ be the composition

$$X_k \xrightarrow{\gamma} X_j \xrightarrow{\alpha} X_i;$$

then, if $S_*^2\alpha \circ S_*^2\gamma$ is similarly defined,

$$S_*^2(\alpha \circ \gamma) = S_*^2\alpha \circ S_*^2\gamma. \tag{2.2}$$

It follows from § 2 in (**1**) that

$$(S_*^2\alpha + S_*^2\beta) \circ (S_*^2\gamma) = S_*^2\alpha \circ S_*^2\gamma + S_*^2\beta \circ S_*^2\gamma, \tag{2.3}$$

and, if $\delta \in G_{h,i}$,

$$S_*^2\delta \circ (S_*^2\alpha + S_*^2\beta) = S_*^2\delta \circ S_*^2\alpha + S_*^2\delta \circ S_*^2\beta. \tag{2.4}$$

Therefore

$$(\alpha+\beta) \circ \gamma = \alpha \circ \gamma + \beta \circ \gamma,$$

$$\delta \circ (\alpha+\beta) = \delta \circ \alpha + \delta \circ \beta. \tag{2.5}$$

If 1_i is the class of the identity map $X_i \to X_i$, we have obviously

$$1_i \circ \alpha = \alpha = \alpha \circ 1_j. \tag{2.6}$$

We have proved

THEOREM 2.7. *If $N \leqslant 2r-2$, $r > 1$, $\Pi(C_r^N)$ is a ringoid when addition is defined by* (2.1), *and multiplication is by composition.*

Though the n-fold suspension homomorphism

$$S_*^n\colon \Pi(C_r^N) \to \Pi(S^nC_r^N)$$

is an isomorphism onto for every n, I do not identify these ringoids,

† I shall frequently omit the subscripts, and write just 0.

since I do not wish to confine myself to the S-theory of the spaces in C_r^N [see (**6**)]. On the other hand, all the results in the present paper have counterparts in S-theory.

3. The ringoid $\Pi(Y^{n+1})$

Let $p > 0$ be an integer and let Y_p^{n+1} be formed by attaching an $(n+1)$-cell to a sphere S^n by a map of degree p; let Y^{n+1} be a collection of spaces Y_p^{n+1}, one for each integer $p = 1, 2, \ldots$; we shall determine the structure of $\Pi(Y^{n+1})$ $(n \geqslant 3)$. The elements can be quickly obtained by obstruction methods, and the additive groups have been determined in (**2**).

We construct the spaces Y_p^{n+1} in a definite way and define three types of maps

$$Y_p^{n+1} \to Y_{pq}^{n+1}, \qquad Y_{pq}^{n+1} \to Y_p^{n+1}, \qquad Y_p^{n+1} \to Y_p^{n+1}.$$

The first maps the $(n+1)$-cell homeomorphically and S^n with degree q on S^n; the second maps S^n homeomorphically and the $(n+1)$-cell with degree q on the $(n+1)$-cell; the third pinches S^n to a point and maps the whole in S^n.

Let S^1 be the unit circle in the Argand diagram, S^n its $(n-1)$th suspension, and let $f_r\colon S^1 \to S^1$ be the map $f_r(e^{i\theta}) = e^{ir\theta}$, a map of degree r. Thus f_1 is the identity map, and

$$f_p \circ f_q = f_{pq}. \tag{3.1}$$

Let $f_r\colon S^n \to S^n$ be the $(n-1)$-fold suspension $S^{n-1}f_r$ of f_r. Then (3.1) is true for these suspended maps.

Let E^{n+1} be a disk bounded by S^n and extend $f_r\colon S^n \to S^n$ to

$$\bar{f}_r\colon (E^{n+1}, S^n) \to (E^{n+1}, S^n)$$

by taking polar coordinates (t, θ) in E^{n+1}, where $0 \leqslant t \leqslant 1$ and

$$(1, \theta) = \theta \in S^n,$$

and setting

$$\bar{f}_r(t, \theta) = (t, f_r\theta).$$

Let $S_p^n = i_p S^n$ be a homeomorph of S^n, and attach a cell e^{n+1} to S_p^n by the map $i_p \circ f_p$, of degree p. Then there is a characteristic map for the $(n+1)$-cell,

$$f_p'\colon (E^{n+1}, S^n) \to (Y_p^{n+1}, S_p^n),$$

such that f_p' maps $E^{n+1} - S^n$ homeomorphically on $Y_p^{n+1} - S_p^n$, and $f_p' \mid S^n = i_p \circ f_p$.

Let homotopy classes $\phi_{pq,p}$, $\psi_{p,pq}$ be defined by

$$\left.\begin{aligned} \phi_{pq,p} &= \{f_{pq}' \circ (f_p')^{-1}\}\colon Y_p^{n+1} \to Y_{pq}^{n+1} \\ \psi_{p,pq} &= \{f_p' \circ \bar{f}_q \circ (f_{pq}')^{-1}\}\colon Y_{pq}^{n+1} \to Y_p^{n+1} \end{aligned}\right\}, \tag{3.2}$$

where the brackets indicate homotopy classes of the contained maps.

Then

$$f'_{pq} \circ (f'_p)^{-1} \mid S^n_p = i_{pq} \circ f_{pq} \circ (f_p)^{-1} \circ i_p^{-1} = i_{pq} \circ f_q \circ (i_p)^{-1},$$

a map of degree q, while

$$f'_p \circ \bar{f}_q \circ (f'_{pq})^{-1} \mid S^n_{pq} = i_p \circ f_p \circ f_q \circ (f_{pq})^{-1} \circ (i_{pq})^{-1} = i_p \circ (i_{pq})^{-1},$$

a homeomorphism.

Let α', μ' be maps

$$Y^{n+1}_p \xrightarrow{\mu'} S^{n+1} \xrightarrow{\alpha'} S^n$$

such that μ' squeezes S^n_p to a point and maps e^{n+1} homeomorphically on $S^{n+1} - \mu'(S^n_p)$, and α' is essential. Define

$$\left.\begin{aligned} \mu_{q,p} &= \{i_q \circ \alpha' \circ \mu'\}\colon Y^{n+1}_p \to Y^{n+1}_q \\ \mu_p &= \mu_{p,p} \end{aligned}\right\} \tag{3.3}$$

Then $\mu_{q,p}$ is inessential if either p or q is odd.

From the definition of $\bar{f}_r$ and (3.1), we see that $\bar{f}_p \circ \bar{f}_q = \bar{f}_{pq}$ and we deduce at once from (3.2)

$$\left.\begin{aligned} &(a) && \phi_{pqr,p} = \phi_{pqr,pq} \circ \phi_{pq,p} \\ &(b) && \psi_{p,pqr} = \psi_{p,pq} \circ \psi_{pq,pqr} \\ &(c) && \phi_{pq,p} \circ \psi_{p,pr} = \psi_{pq,pqr} \circ \phi_{pqr,pr} \end{aligned}\right\}. \tag{3.4}$$

Now let $(p,q) = \text{G.C.D.}(p,q)$, and define $k = pq/(p,q)$. Then, again from the definitions, we have

$$\left.\begin{aligned} &(a) && \mu_{q,p} = \psi_{q,k} \circ \mu_k \circ \phi_{k,p} \\ &(b) && \mu_{r,pq} \circ \phi_{pq,p} = \mu_{r,p} = \psi_{r,rq} \circ \mu_{rq,p} \\ &(c) && \phi_{pq,p} \circ \mu_{p,r} = q\mu_{pq,r} \\ &(d) && \mu_{p,q} \circ \psi_{q,qr} = r\mu_{p,qr} \\ &(e) && \mu_{p,q} \circ \mu_{q,r} = 0 \end{aligned}\right\}. \tag{3.5}$$

Let $1_p = \phi_{p,p} = \psi_{p,p}$ be the class of the identity map. Notice that, using (3.5a), we can replace any expression in the $\mu_{q,p}$, $\phi_{pq,p}$, $\psi_{p,pq}$ by one involving μ_p, $\phi_{pq,p}$, $\psi_{p,pq}$ only. However, it is convenient to retain the redundant $\mu_{q,p}$.

By a *set of generators* for a ringoid we mean a collection of elements such that any element of the ringoid is a sum of products of these generators; for example, we could take all the elements of the ringoid as a set of generators. My principal theorem† is

THEOREM 3.6. *The ringoid* $\Pi(Y^{n+1})$ $(n \geqslant 3)$ *is generated by* 1_p, μ_p, $\phi_{pq,p}$, $\psi_{p,pq}$, *for all* $p > 0$, $q > 1$. *All relations are implied by relations* (3.4), (3.5), *and* (3.7) *below, in which* $\mu_{q,p}$ *is defined by* (3.5a).

† This gives both the additive and the multiplicative structure of the ringoid.

$$\left.\begin{array}{ll} (a) & 2\mu_{q,p} = 0;\ \mu_{q,p} = 0 \text{ if and only if either } p \text{ or } q \text{ is odd} \\ (b) & p1_p = \frac{1}{2}p(p-1)\mu_p \\ (c) & \psi_{p,pq} \circ \phi_{pq,p} = q1_p + \frac{1}{4}pq(p-1)(q-1)\mu_p. \end{array}\right\} \quad (3.7)$$

Except for (3.7 *b*), (3.7 *c*), the theorem is a straightforward application of obstruction theory; (3.7 *b*) is, effectively, proved in (**2**), and we shall deduce (3.7 *c*) from (3.7 *b*), using the fact that the left-hand side is a suspension of a class of maps $Y_p^2 \to Y_p^2$.

4. Proof of (3.7)

To prove (3.7 *a*), let $g: Y_p^{n+1} \to Y_q^{n+1}$ be the constant map which agrees on S_p^n with $i_q \circ \alpha' \circ \mu'$: then the separation cochain $d = d(i_q \circ \alpha' \circ \mu', g)$ has value† $\{i_q \circ \alpha'\} \in \pi_{n+1}(Y_q^{n+1})$ on the $(n+1)$-cell. Since $n \geqslant 3$, $d = 0$ if q is odd, and is a coboundary if p is odd; in either case, $i_q \circ \alpha' \circ \mu'$ is inessential. If p, q are both even, the cohomology class of d is non-zero, so that $\mu_{q,p} \neq 0$. But then

$$2\mu_{q,p} = 2\{i_q\} \circ \{\alpha'\} \circ \{\mu'\} = \{i_q\} \circ (2\{\alpha'\}) \circ \{\mu'\} = 0,$$

since $2\{\alpha'\} = 0$.

To prove (3.7 *b*), observe that in (**2**) it is shown that, for some extension g of the identity map $S_p^n \to Y_p^{n+1}$,

$$p\{g\} = \lambda\{i_q \circ \alpha' \circ \mu'\} = \lambda\mu_{q,p},$$

where $\lambda \equiv \frac{1}{2}p(p-1) \pmod 2$. We have seen that $p\mu_p = 0$ for all p, so that

$$p(1_p + \mu_p) = p1_p.$$

However, there are at most two classes of extensions of the identity map i_p, and they are 1_p, $1_p + \mu_p$ if these are distinct (i.e. if p is even: if p is odd, there is only one class $\{g\} = 1_p$). Therefore $p1_p$ has the required value.

We now proceed to the proof of (3.7 *c*). The composition $\psi_{p,pq} \circ \phi_{pq,p}$ is, by definition, the class of

$$f'_p \circ \bar{f}_q \circ (f'_{pq})^{-1} \circ f'_{pq} \circ (f'_p)^{-1} = f'_p \circ \bar{f}_q \circ (f'_p)^{-1}.$$

This is a cellular map (if $S_p^n = e^0 \cup e^n$), mapping S_p^n with degree $pqp^{-1} = q$ on itself. Therefore $\psi_{p,pq} \circ \phi_{pq,p}$ and $q1_p$ contain maps which agree on S_p^n, and so, for some integer $D(p,q) = 0, 1$,

$$\psi_{p,pq} \circ \phi_{pq,p} = q1_p + D(p,q)\mu_p. \quad (4.1)$$

† Brackets indicate homotopy class.

When p is odd, $\mu_p = 0$, and we may take $D(p,q) = 0$ in this case. We suppose p even unless otherwise stated.

Expand $\psi_{p,pqr} \circ \phi_{pqr,p}$ in two ways by the relations (3.4):

$$(a)\quad \psi_{p,pqr} \circ \phi_{pqr,p} = \psi_{p,pq} \circ \psi_{pq,pqr} \circ \phi_{pqr,pq} \circ \phi_{pq,p},$$

$$(b)\quad \psi_{p,pqr} \circ \phi_{pqr,p} = \psi_{p,pq} \circ \psi_{pq,pqr} \circ \phi_{pqr,pr} \circ \phi_{pr,p}$$
$$= \psi_{p,pq} \circ \phi_{pq,p} \circ \psi_{p,pr} \circ \phi_{pr,p}.$$

On expanding these by (4.1) and (3.5) we obtain

$$(a')\quad \psi_{p,pqr} \circ \phi_{pqr,p} = \psi_{p,pq} \circ (r1_{pq} + D(pq,r)\mu_{pq}) \circ \phi_{pq,p}$$
$$= r\psi_{p,pq} \circ \phi_{pq,p} + D(pq,r)\mu_p$$
$$= qr1_p + rD(p,q)\mu_p + D(pq,r)\mu_p,$$

$$(b')\quad \psi_{p,pqr} \circ \phi_{pqr,p} = (q1_p + D(p,q)\mu_p) \circ (r1_p + D(p,r)\mu_p)$$
$$= qr1_p + rD(p,q)\mu_p + qD(p,r)\mu_p.$$

Comparing these we have, since μ_p is of order 2,

$$D(pq,r) \equiv qD(p,r) \pmod 2. \tag{4.2}$$

This has two consequences: if q is even (so that pq is divisible by 4) then $D(pq,r) \equiv 0 \pmod 2$ for every r. Also, if we take $p = 2$ and q odd, then $D(2q,r) \equiv D(2,r) \pmod 2$ for every r. Thus we have

LEMMA 4.3. *$D(p,q)\mu_p = \frac{1}{2}p(p-1)D(2,q)\mu_p$ for every p, q.*

This form includes in fact odd values of p, which reduce it to triviality. We must now compute $D(2,q)$: it does not appear possible to deduce this from the previous relations.

Observe that $\psi_{p,pq} \circ \phi_{pq,p}$ is the class of $f'_p \circ \bar{f}_q \circ (f'_p)^{-1}$, which is the $(n-1)$-fold suspension of the similar map

$$f'_p \circ \bar{f}_q \circ (f'_p)^{-1} \colon Y^2_p \to Y^2_p.$$

Therefore $q1_2 + D(2,q)\mu_2$ is an $(n-1)$-fold suspension element. We prove

LEMMA 4.4. *The only classes of $(n-1)$-fold suspensions of maps $Y^2_2 \to Y^2_2$ are the identity element 1_2 and the zero element ($n \geqslant 2$).*

Let $1, 0 \colon Y^{n+1}_2 \to Y^{n+1}_2$ be, respectively, the identity map and a constant map, for $n \geqslant 1$. Then $S1 = 1$, and we may so choose 0 that $S0 = 0$. Given a map $g \colon Y^2_2 \to Y^2_2$, we may so deform it that $g \mid S^1_2 = t \mid S^1_2$, where $t = 0$ or 1. Then g, t have a separation cochain $d(g,t)$, which we identify with its value in $\pi_2(Y^2_2)$ on e^2. A simple argument shows that

$$d(S^k g, S^k t) = S^k_* d(g,t),$$

where we again identify the cochain with its value on the $(k+2)$-cell.

Therefore, if we prove

LEMMA 4.5. *$S_*\pi_2(Y_2^2) \subset 2\pi_3(Y_2^3)$, and, for $n > 2$, $S_*^{n-1}\pi_2(Y_2^2) = 0$,*

we shall be able to deduce Lemma 4.4: for $S_* d(g,t)$ is a coboundary or zero, and $S_*^{n-1} d(g,t) = 0$ for $n > 2$. Thus, for $n \geqslant 2$, $S^{n-1}g$ and $S^{n-1}t$ are homotopic.

Let† x be the non-zero element of $\pi_1(Y_2^2)$, and let ξ be a generator of $\pi_2(Y_2^2) \approx \pi_2(S^2)$. Then, if . indicates the operation of π_1 on π_2,

$$x.\xi = -\xi,$$

so that
$$2\xi = \xi - x.\xi = -[x,\xi],$$

the Whitehead product. Now, according to (2.27) in (**7**), if $\beta \in \pi_p(X)$, $\gamma \in \pi_q(X)$,

$$S_*[\beta,\gamma] = 0 \in \pi_{p+q}(SX),$$

whence $2S_*\xi = 0$. Since $\pi_3(Y_2^3)$ is cyclic of order 4, $S_*\xi \in 2\pi_3(Y_2^3)$; and it follows that $S_*^2\xi = 0$. This proves (4.5) and completes the proof of (4.4).

Now $q1_2 + D(2,q)\mu_2 = 1_2$ or 0 since it is a suspension element. By (3.7 *b*), $2\,1_2 = \mu_2$; therefore, if $q \equiv 0, 1 \pmod 4$,

$$q1_2 + D(2,q)\mu_2 = \epsilon 1_2 + D(2,q)\mu_2,$$

where $\epsilon = 0$ or 1: in this case $D(2,q) = 0$. If $q \equiv 2, 3 \pmod 4$,

$$q1_2 + D(2,q)\mu_2 = \epsilon 1_2 + D(2,q)\mu_2 + \mu_2,$$

where $\epsilon = 0$ or 1: in this case, $D(2,q) = 1$. Therefore

$$D(2,q) \equiv \tfrac{1}{2}q(q-1) \pmod 2,$$

and so
$$D(p,q)\mu_p = \tfrac{1}{4}pq(p-1)(q-1)\mu_p$$

in all cases. This completes the proof of relations (3.7).

5. Proof of Theorem 3.6

The relations (3.7 *b*), (3.5) imply

LEMMA 5.1. *$p\phi_{pq,p} = \tfrac{1}{2}pq(p-1)\mu_{pq,p}$, $p\psi_{p,pq} = \tfrac{1}{2}pq(p-1)\mu_{p,pq}$,*

$$d\psi_{q,k} \circ \phi_{k,p} = \tfrac{1}{2}k(k-1)\mu_{q,p},$$

where $d = $ G.C.D.(p,q), and $k = pq/d$.

For, by (3.7 *b*) and (3.5 *c*),

$$p\phi_{pq,p} = \phi_{pq,p} \circ (p1_p) = \tfrac{1}{2}p(p-1)\phi_{pq,p} \circ \mu_p = \tfrac{1}{2}pq(p-1)\mu_{pq,p};$$

the second relation follows similarly from (3.7 *b*) and (3.5 *d*).

† This proof was kindly suggested to me by Professor J. H. C. Whitehead.

Now integers a, b exist such that $ap+bq = d$; therefore

$$\begin{aligned} d\psi_{q,k} \circ \phi_{k,p} &= b(q\psi_{q,k}) \circ \phi_{k,p} + a\psi_{q,k} \circ (p\phi_{k,p}) \\ &= \tfrac{1}{2}bq(q-1)(k/q)\mu_{q,k} \circ \phi_{k,p} + \tfrac{1}{2}ap(p-1)(k/p)\psi_{q,k} \circ \mu_{k,p} \\ &= \tfrac{1}{2}k(bq-b+ap-a)\mu_{q,p} \\ &= \tfrac{1}{2}k(d-a-b)\mu_{q,p}. \end{aligned}$$

If either p or q is odd, $\mu_{q,p} = 0 = \frac{1}{2}k(k-1)\mu_{q,p}$. If both p, q are even and $k \equiv 0 \pmod 4$, both $\frac{1}{2}k(d-a-b)$ and $\frac{1}{2}k(k-1)$ are even. If p and q are even but $k \equiv 2 \pmod 4$, then $d \equiv 2 \pmod 4$, and $(a+b) \equiv 1 \pmod 2$, since this situation occurs only when $p \equiv q \equiv 2 \pmod 4$. In this case, both $\frac{1}{2}k(d-a-b)$ and $\frac{1}{2}k(k-1)$ are odd. This proves the lemma.

We now exhibit a canonical form for an element of the ringoid.

LEMMA 5.2. *Every element of* $\Pi(Y^{n+1})$ *has a unique expression of the form* $\alpha = m\psi_{q,k} \circ \phi_{k,p} + \epsilon\mu_{q,p}$, *where* $k = pq/(p,q)$, $0 \leqslant m < (p,q)$, *and* $\epsilon = 0, 1$ *if* p, q *are both even*; $\epsilon = 0$ *otherwise*.

Recall that Y^{n+1}_p has a cell structure $e^0 \cup e^n \cup e^{n+1}$. Any element $\alpha \in \Pi(Y^{n+1})$ can be represented by a cellular map $g' : Y^{n+1}_p \to Y^{n+1}_q$. Then $g' \circ i_p : S^n \to Y^{n+1}_q$ maps S^n with some degree r' on S^n_q (if S^n_q is oriented by means of the map i_q). If $r' = 0$, we can deform g' so that $g' \circ i_p$ is a constant map; otherwise we can deform g' into a cellular map g such that $g \circ i_p = i_q \circ f_r$, for some *positive* integer r. Here f_r is a map of the type described in § 3.

In the first case, we may suppose that g' and $i_q \circ \alpha' \circ \mu'$ agree on S^n_p, and so possess a separation cochain d. If d is zero or a coboundary, these are homotopic and $\alpha = \mu_{q,p}$. If d is non-zero, we may compare g' with the constant map: the separation cochain must vanish, and we deduce that $\alpha = 0$.

We now examine the map g: since $r > 0$, consideration of the obstruction to the extension of $g \mid S^n_p$ over the whole shows that q divides pr. Let $k = pq/(p,q)$: then $k/p = q/(p,q)$ divides r. Therefore, for some integer m, $pr = mk$. We compare α with $\psi_{q,mk} \circ \phi_{pr,p}$, which latter has a representative map

$$h = f'_q \circ \bar{f}_{mk/q} \circ (f'_{mk})^{-1} \circ f'_{pr} \circ (f'_p)^{-1} = f'_q \circ \bar{f}_{mk/q} \circ (f'_p)^{-1}.$$

Then g, h agree on S^n_p, and possess a separation cochain d. Then it is easy to show that $\alpha = \psi_{q,mk} \circ \phi_{mk,p}$ or $\alpha = \psi_{q,mk} \circ \phi_{mk,p} + \mu_{q,p}$ according as the cohomology class of d is zero or not.

Now from (3.4 *a*), (3.4 *b*), (3.7 *c*), for some $\epsilon' = 0, 1$,

$$\psi_{q,mk} \circ \phi_{mk,p} = m\psi_{q,k} \circ \phi_{k,p} + \epsilon'\mu_{q,p}.$$

Therefore, in all cases discussed, for some integer $m \geqslant 0$, and some $\epsilon = 0, 1$,

$$\alpha = m\psi_{q,k} \circ \phi_{k,p} + \epsilon\mu_{q,p}. \tag{5.3}$$

However, for some s, m', where $0 \leqslant m' < (p,q)$,

$$m = m' + s(p,q).$$

Then it follows from this, (5.3), and Lemma 5.1 that α can be put in the required form.

We must now show that this is unique: clearly, it is sufficient to consider $0 = m\psi_{q,k} \circ \phi_{k,p} + \epsilon\mu_{q,p}$. This element can be represented by a cellular map g such that $g \circ i_p$ maps S^n with degree $mk/p = mq/(p,q)$ on S^n_q. Therefore g is essential unless $m = 0$, or (p,q) divides m, which last is impossible. Therefore $m = 0$. Also, if p, q are both even, $\mu_{q,p} \neq 0$, so $\epsilon = 0$.

The proof of Theorem 3.6 is now simple: by (3.5*a*),

$$\mu_{q,p} = \psi_{q,k} \circ \mu_k \circ \phi_{k,p},$$

and by Lemma 5.2 any element can be put in the form

$$\alpha = \psi_{q,k} \circ (m1_k + \epsilon\mu_k) \circ \phi_{k,p}.$$

Therefore $\Pi(Y^{n+1})$ is generated by the stated elements.

We now prove that, if α is any expression in the generators, it can be reduced to the canonical form, as in (5.2), using the listed relations. If $\alpha = 0$, the canonical form is identically zero, so that we can deduce that every relation is a consequence of relations (3.4), (3.5), (3.7).

If α is given as a sum of products of the generators, we say that α is a sum of *terms*, and the generators in each term are called its *factors*. If α is given in any other way, it can be written as a sum of products, using the distributive laws.

If a term of α does not contain a factor $\mu_{i,j}$ (for any i, j), it can be transformed by relations (3.4) into a product in which all the ψ's are to the left of all the ϕ's; therefore, by (3.4) again, it can be put in the form $\psi_{q,s} \circ \phi_{s,p}$, where both p and q divide s. If a term does contain a factor $\mu_{i,j}$, it can be reduced to a multiple of $\mu_{q,p}$ by using relations (3.5) (it may be a zero multiple).

Consider the terms of the first sort: since p, q divide s, so does $k = pq/(p,q)$, and, for some m, $s = mk$. Using (3.4), (3.7*c*), we get

$$\begin{aligned}
\psi_{q,s} \circ \phi_{s,p} &= \psi_{q,k} \circ \psi_{k,mk} \circ \phi_{mk,k} \circ \phi_{k,p} \\
&= \psi_{q,k} \circ (m1_k + \epsilon\mu_k) \circ \phi_{k,p} \\
&= m\psi_{q,k} \circ \phi_{k,p} + \epsilon\mu_{q,p},
\end{aligned}$$

for some $\epsilon = 0, 1$. Adding all the terms together, and using Lemma 5.1 and (3.7 a), we find that we have expressed α in canonical form. This completes the proof of Theorem 3.6.

6. On $\Pi(C_n^{n+1})$

Let C_n^{n+1} $(n \geqslant 3)$ be a collection of finite $(n-1)$-connected CW-complexes of dimension not exceeding $n+1$. Since they are A_n^1-polyhedra, we can suppose without loss of generality that they are in the normal form for A_n^2-polyhedra described in (**4**). That is, if by a *cluster* of spaces K_i we mean a space $K = \bigcup K_i$ such that pairs of distinct spaces K_i, K_j meet in but one point, common to them all, then $K \in C_n^{n+1}$ is a cluster of spaces of the types S^n, S^{n+1}, Y_p^{n+1}.

Let $\{A; B\}$ denote the additive group of homotopy classes of maps $A \to B$; let K be a cluster of spaces K_i, and let σ_i, τ_i be respectively the classes of the retractions and injections

$$\sigma_i \colon K \to K_i; \qquad \tau_i \colon K_i \to K.$$

Then it follows from Theorem 7.41 in (**1**) that, if K is in some C_n^{2n-2}, there is an isomorphism

$$\tau^* \colon \{K; X\} \approx \Sigma_i^* \{K_i; X\}$$

where Σ_i^* denotes the strong sum and $\tau^*(\alpha)$ has coordinate $\alpha \circ \tau_i$ in $\{K_i; X\}$.

Let K be a finite cluster of spaces K_i, let $K = X$, and let 1, 1_i be the classes of the identity maps $K \to K$, $K_i \to K_i$. Since $\sigma_i \circ \tau_i = 1_i \colon K_i \to K_i$, $\sigma_i \circ \tau_j = 0$ $(i \neq j)$,

$$\tau^*(\Sigma_i \tau_i \circ \sigma_i) = \Sigma_i^*(\tau_i \circ \sigma_i \circ \tau_i) = \Sigma_i^*(\tau_i) = \tau^*(1).$$

Therefore† $\Sigma_i \tau_i \circ \sigma_i = 1$ in $\{K; K\}$.

Let K be a finite cluster of K_i, K' a finite cluster of K_j', and define a homomorphism

$$\theta \colon \Sigma_{i,j}^* \{K_i; K_j'\} \to \{K; K'\}$$

by

$$\theta\{\Sigma_{i,j}^*(\alpha_{j,i})\} = \Sigma_{i,j} \tau_j' \circ \alpha_{j,i} \circ \sigma_i, \qquad \alpha_{j,i} \in \{K_i; K_j'\}, \tag{6.1}$$

where the summation on the left is direct summation,‡ that on the right is summation in $\{K; K'\}$, and where τ_j' has a similar meaning to τ_i. Then we have

LEMMA 6.2. *θ is an isomorphism onto.*

For let σ_j' have a similar meaning to σ_i, and define

$$\nu(\alpha) = \Sigma_{i,j}^*(\sigma_j' \circ \alpha \circ \tau_i), \qquad \alpha \in \{K; K'\}.$$

† This has also been proved by J. H. C. Whitehead.

‡ Since the number of groups is finite, the strong sum is the direct sum.

Then $$\nu\theta(\Sigma^*_{i,j}\alpha_{j,i}) = \Sigma^*_{i,j}\alpha_{j,i},$$

and
$$\begin{aligned}\theta\nu(\alpha) &= \Sigma_{i,j}\tau'_j \circ \sigma'_j \circ \alpha \circ \tau_i \circ \sigma_i \\ &= (\Sigma_j \tau'_j \circ \sigma'_j) \circ \alpha \circ (\Sigma_i \tau_i \circ \sigma_i) \\ &= \alpha,\end{aligned}$$

by the remark above. Therefore θ has a two-sided inverse ν, and we may use it to identify the groups.

Notice that, if K' is a finite cluster of spaces and K is an arbitrary cluster, we can set up an isomorphism similar to θ, using τ^*. We can extend this further if K is a cluster of finite complexes and K' an arbitrary cluster.

Let K, K', K'' be finite clusters of subcomplexes K_i, K'_j, K''_k: then the direct-sum decomposition leads to a convenient formula for multiplication, for we can verify

$$(\Sigma^*_{j,k}\beta_{k,j}) \circ (\Sigma^*_{i,j}\alpha_{j,i}) = \Sigma^*_{i,k}(\Sigma_j \beta_{k,j} \circ \alpha_{j,i}). \tag{6.3}$$

Alternatively, one can express these results in matrix form, with the usual definition of addition and multiplication.

These formulae show that the ringoid of C^{n+1}_n, a collection of finite A^1_n-polyhedra, can be computed from the ringoids of the collections which consist of triples of spaces (X, Y, Z), where X, Y, Z are, separately, S^n, S^{n+1}, or in Y^{n+1}. If they are all in Y^{n+1}, the ringoid can be obtained from Theorem 3.6; there is no difficulty in computing them in other cases, and they can be deduced from the results below.

7. The ringoid $\Pi(Y^{n+1}, S^m)$; homotopy groups

Let $\max(m, n+1) \leqslant \min(2m-2, 2n-2)$, and let the collection Y^{n+1} be augmented by a sphere S^m: we shall compute the ringoid of the augmented collection, $\Pi(Y^{n+1}, S^m)$. It contains the subringoid $\Pi(S^m)$ isomorphic to the ring of integers and the ringoid $\Pi(Y^{n+1})$ already discussed. Our problem is to compute the groups

$$\pi_m(Y^{n+1}_p) \approx \{S^m; Y^{n+1}_p\} \quad (m \leqslant 2n-2),$$

the groups
$$\{Y^{n+1}_p; S^m\} \quad (n \leqslant 2m-3),$$

and the operations of $\Pi(Y^{n+1})$ on the left and right respectively. The operations of the ring $\Pi(S^m)$ are determined by the distributive laws of multiplication. The groups $\{Y^{n+1}_p; S^m\}$ are given by formulae in (**2**). Our results will be expressed in terms of the homotopy groups of spheres. We first calculate the homotopy groups $\pi_m(Y^{n+1}_p)$, and the effect of left multiplication by elements of $\Pi(Y^{n+1})$.

It follows from (**4**) that, if p is a product of mutually prime numbers $p_1,\ldots, p_T$, then Y_p^{n+1} is of the homotopy type of a cluster of spaces $Y_{p_1}^{n+1},\ldots, Y_{p_T}^{n+1}$. Let K be such a cluster, $\sigma_i\colon K \to Y_{p_i}^{n+1}$ the class of the retraction; then

$$\sigma = \Sigma_i \phi_{p,p_i} \circ \sigma_i\colon K \to Y_p^{n+1}$$

is a class of homotopy equivalences. For, if $f \in \sigma$, u generates $H_n(S^n)$, $u_i = i_{p_i *}(u)$ generates $H_n(Y_{p_i}^{n+1})$, and, if we identify $H_n(K)$ and $\Sigma_i^* H_n(Y_{p_i}^{n+1})$ $(n > 0)$,

$$f_*(\Sigma_i^* t_i u_i) = \Sigma_i t_i (p/p_i) i_{p*}(u),$$

for all integers t_i.

Now the p_i are mutually prime, and so therefore are the integers $p/p_1,\ldots, p/p_T$. Choose t_i so that $\Sigma_i t_i(p/p_i) = 1$: we deduce that f_* is onto. Also, if $\Sigma_i t_i(p/p_i) \equiv 0 \pmod p$, then each $t_i \equiv 0 \pmod{p_i}$; therefore f_* is an isomorphism onto. Hence f induces an isomorphism of the homology groups, and so is a homotopy equivalence (**10**).

A homotopy class of maps $\psi\colon X \to Y$ induces a homomorphism $\psi\colon \pi_m(X) \to \pi_m(Y)$ such that $\psi(\alpha) = \alpha \circ \psi$. I shall write $\psi \circ \pi_m(X)$ for the image under ψ. In § 6 we identified $\pi_m(K)$ with $\Sigma_i^* \pi_m(Y_{p_i}^{n+1})$; then the isomorphism $\sigma\colon \Sigma_i^* \pi_m(Y_{p_i}^{n+1}) \approx \pi_m(Y_p^{n+1})$ is such that

$$\sigma \circ (\Sigma_i^* \alpha_i) = \Sigma_i \phi_{p,p_i} \circ \alpha_i.$$

Thus we have

THEOREM 7.1. *If $m \leqslant 2n-2$, and p is a product $p_1 \ldots p_T$ of mutually prime numbers, then*

$$\pi_m(Y_p^{n+1}) = \Sigma_i \phi_{p,p_i} \circ \pi_m(Y_{p_i}^{n+1}),$$

where each ϕ_{p,p_i} is an isomorphism onto a direct summand.

The characteristic map $f'_p\colon (E^{n+1}, S^n) \to (Y_p^{n+1}, S_p^n)$ of the $(n+1)$-cell induces for $m \leqslant 2n-1$ an isomorphism [(**8**), (**5**)]

$$f'_p\colon \pi_m(E^{n+1}, S^n) \approx \pi_m(Y_p^{n+1}, S_p^n)$$

such that the following diagram is commutative:

$$\begin{array}{ccccccccc} \to \pi_m(S^n) & \xrightarrow{i_p} & \pi_m(Y_p^{n+1}) & \xrightarrow{j_p} & \pi_m(Y_p^{n+1}, S_p^n) & \xrightarrow{\partial_p} & \pi_{m-1}(S^n) \to \\ & & & & \uparrow f'_p & & \uparrow f_p \\ & & & & \pi_m(E^{n+1}, S^n) & \xrightarrow{\partial} & \pi_{m-1}(S^n) \end{array}$$

In this, i_p and f_p are the homomorphisms induced by the maps i_p, f_p, and the horizontal sequence is equivalent to the exact homotopy sequence of the pair (Y_p^{n+1}, S_p^n).

This can therefore be replaced by an exact sequence, beginning with $i_p\colon \pi_{2n-1}(S^n) \to \pi_{2n-1}(Y_p^{n+1})$,

$$\to \pi_m(S^n) \xrightarrow{f_p} \pi_m(S^n) \xrightarrow{i_p} \pi_m(Y_p^{n+1}) \xrightarrow{j'_p} \pi_{m-1}(S^n) \xrightarrow{f_p} \pi_{m-1}(S^n) \to, \quad (7.2)$$

where $$j'_p = \partial \circ (f'_p)^{-1} \circ j_p. \tag{7.3}$$

Notice that $S_* \circ j'_p \colon \pi_m(Y_p^{n+1}) \to \pi_m(S^{n+1})$ is the homomorphism induced by the map $\mu' \colon Y_p^{n+1} \to S^{n+1}$ if this map is suitably chosen in the paragraph containing (3.3) above.

Now, if G is an additive abelian group, an integer p determines a homomorphism $p \colon G \to G$ such that $p(g) = pg$. Let ${}_pG$ be the kernel of this, the subgroup of elements whose order divides p, and let G_p be the co-kernel, the factor group $G/{}_pG$. Let $\kappa_p \colon G \to G_p$ be the natural projection. If there is an exact sequence

$$0 \to G_2 \xrightarrow{i} E \xrightarrow{j} {}_2H \to 0,$$

it is said that E is an *extension* of G_2 by ${}_2H$. Then it is shown in (**9**) that E is uniquely characterized by the homomorphism $e \colon {}_2H \to G_2$ such that $e(h) = i^{-1}\{2(j^{-1}(h))\}$: that is, given any $e \colon {}_2H \to G_2$, there is a unique equivalence class of extensions E' with this homomorphism.

THEOREM 7.4. *$\pi_m(Y_p^{n+1})$ is an extension of $\pi_m(S^n)_p$ by ${}_p\pi_{m-1}(S^n)$, for any p, any $m \leqslant 2n-2$. If $p \neq 2$ is a power of a prime, the extension is the direct sum. If $p = 2$, and α^n generates $\pi_{n+1}(S^n)$, the extension is determined by the homomorphism e, where*

$$e(\theta) = \kappa_2(\alpha^n \circ S_* \theta) \in \pi_m(S^n)_2.$$

The first part is well known, and follows from the exactness of (7.2). The method of determining the extension is essentially that used in (**3**) in the case $m = n+2$, $p = 2$.

Let p be a power of a prime; then, by (3.7 *b*), $p\mathbf{1}_p = 0$ if $p \neq 2$. Hence, for any $\gamma \in \pi_m(Y_p^{n+1})$ $(p \neq 2)$,

$$p\gamma = p\mathbf{1}_p \circ \gamma = 0.$$

In particular, if p is a prime other than 2, every element of $\pi_m(Y_p^{n+1})$ is of order p, and the extension is the direct sum. On the other hand, if $p = 2$, $2\,\mathbf{1}_2 = \mu_2$, where μ_2 is the homotopy class of the map $i_2 \circ \alpha' \circ \mu'$. Then, for $\gamma \in \pi_m(Y_2^{n+1})$,

$$\begin{aligned} 2\gamma = 2\,\mathbf{1}_2 \circ \gamma &= \{i_2\} \circ \{\alpha'\} \circ \{\mu'\} \circ \{\gamma\} \\ &= \{i_2\} \circ \alpha^n \circ S_* j'_2(\gamma) \\ &= i_2(\alpha^n \circ S_* j'_2(\gamma)), \end{aligned} \tag{7.5}$$

and i_2 is the natural projection κ_2.

If p is a higher power of any prime, we must proceed more indirectly. We have homomorphisms

$$\phi_{pq,p} \colon \pi_m(Y_p^{n+1}) \to \pi_m(Y_{pq}^{n+1}), \qquad \psi_{p,pq} \colon \pi_m(Y_{pq}^{n+1}) \to \pi_m(Y_p^{n+1}),$$

and prove

LEMMA 7.6.

$$j'_{pq} \circ \phi_{pq,p} = j'_p, \qquad j'_p \circ \psi_{p,pq} = f_q \circ j'_{pq},$$
$$\phi_{pq,p} \circ i_p = i_{pq} \circ f_q, \qquad \psi_{p,pq} \circ i_{pq} = i_p.$$

Let $\{\}$ denote the homotopy class of the contained map, so that $i_p = \{i_p\}$, $f_p = \{f_p\}$. Then, for the last two formulae, observe, since $f'_p \mid S^n = i_p \circ f_p$, that

$$\begin{aligned}
\phi_{pq,p} \circ i_p &= \{f'_{pq} \circ (f'_p)^{-1} \circ i_p\} \\
&= \{i_{pq} \circ f_{pq} \circ (f_p)^{-1}\} \\
&= \{i_{pq} \circ f_q\} = i_{pq} \circ f_q, \\
\psi_{p,pq} \circ i_{pq} &= \{f'_p \circ \bar{f}_q \circ (f'_{pq})^{-1} \circ i_{pq}\} \\
&= \{i_p \circ f_p \circ f_q \circ (f_{pq})^{-1}\} \\
&= i_p.
\end{aligned}$$

Now $f'_{pq} \circ (f'_p)^{-1}$ is a cellular map and determines a map

$$h\colon (Y^{n+1}_p, S^n_p) \to (Y^{n+1}_{pq}, S^n_{pq})$$

with the same values; clearly

$$h_*\colon \pi_m(Y^{n+1}_p, S^n_p) \to \pi_m(Y^{n+1}_{pq}, S^n_{pq})$$

is such that

$$\left.\begin{aligned} &(a)\ \ h_* \circ f'_p = f'_{pq} \\ &(b)\ \ h_* \circ j_p = j_{pq} \circ \phi_{pq,p} \end{aligned}\right\}.$$

Since f'_p, f'_{pq} are isomorphisms onto (in the range considered), so is h_*. Since $j'_p = \partial \circ (f'_p)^{-1} \circ j_p$, we have

$$\begin{aligned}
j'_{pq} \circ \phi_{pq,p} &= \partial \circ (f'_{pq})^{-1} \circ j_{pq} \circ \phi_{pq,p} \\
&= \partial \circ (h_* \circ f'_p)^{-1} \circ h_* \circ j_p \\
&= \partial \circ (f'_p)^{-1} \circ j_p \\
&= j'_p.
\end{aligned}$$

Similarly $\psi_{p,pq}$ is represented by the cellular map $f'_p \circ \bar{f}_q \circ (f'_{pq})^{-1}$ which determines

$$k\colon (Y^{n+1}_{pq}, S^n_{pq}) \to (Y^{n+1}_p, S^n_p),$$

with the same values, such that the induced homomorphism k_* satisfies

$$k_* \circ j_{pq} = j_p \circ \psi_{p,pq}.$$

Thus

$$\begin{aligned}
j'_p \circ \psi_{p,pq} &= \partial \circ (f'_p)^{-1} \circ j_p \circ \psi_{p,pq} \\
&= \partial \circ (f'_p)^{-1} \circ k_* \circ j_{pq} \\
&= \partial \circ \bar{f}_q \circ (f'_{pq})^{-1} \circ j_{pq} \\
&= f_q \circ \partial \circ (f'_p)^{-1} \circ j_{pq} \\
&= f_q \circ j'_{pq}
\end{aligned}$$

since $\partial \circ \bar{f}_q = f_q \circ \partial$. This proves the lemma.

Since $\pi_{m-1}(S^n)$ is finitely generated, ${}_p\pi_{m-1}(S^n)$ is the direct sum of cyclic groups of orders dividing p, which we suppose is a power of a prime. Choose a set of generators of these cyclic summands, and let θ be one, of order q, say, a divisor of p. Then $\theta \in {}_q\pi_{m-1}(S^n)$, and there exists a $\bar{\gamma} \in \pi_m(Y_q^{n+1})$ such that $j_q''(\bar{\gamma}) = \theta$; we have seen that $q\bar{\gamma} = 0$ unless $q = 2$, when $2\bar{\gamma}$ is given by (7.5). Let $\gamma = \phi_{p,q} \circ \bar{\gamma}$; by Lemma 7.6,

$$j_p''(\gamma) = j_q''(\bar{\gamma}) = \theta,$$

$$q\gamma = \phi_{p,q} \circ (q\bar{\gamma}) = 0 \quad \text{unless } q = 2.$$

When $q = 2$, by (7.5),

$$\begin{aligned} 2\gamma &= \phi_{p,2} \circ i_2 \circ \tilde{\alpha}^n \circ S_* \theta \\ &= i_p(\tfrac{1}{2}p\alpha^n \circ S_* \theta). \end{aligned}$$

Now α^n is of order 2, so that either $p = 2$, or $\frac{1}{2}p\alpha^n = 0$. Therefore we have proved

$$q\gamma = 0, \quad \text{unless } p = q = 2. \tag{7.7}$$

Here p is a power of a prime. This, in conjunction with (7.5), completes the proof of Theorem 7.4.

Now we have shown that, if $p \neq 2$ is a power of a prime,

$$\pi_m(Y_p^{n+1}) \approx \pi_m(S^n)_p + {}_p\pi_{m-1}(S^n);$$

however, this decomposition is not natural, in the sense that j_p' has not a natural right inverse (compare the natural summands in Theorem 7.1). This leads to awkwardness in the description of $\Pi(Y^{n+1}, S^m)$.

We now compute the operations of $\Pi(Y^{n+1})$ on these homotopy groups. It is clearly sufficient to compute the operations of the generators given in Theorem 3.6.

Obviously 1_p is a left identity on $\pi_m(Y_p^{n+1})$, and we have seen, for $\gamma \in \pi_m(Y_p^{n+1})$, that

$$\mu_p \circ \gamma = i_p \circ \alpha^n \circ S_* j_p''(\gamma).$$

To determine the operation of the other elements, we must give a more detailed description of the groups.

Since $\pi_{m-1}(S^n)$ is finitely generated, it can be expressed as a direct sum of cyclic groups whose orders are powers of primes or are infinite. In such a decomposition, choose a generator for each of the finite cyclic groups, and order the generators. Then the subgroup of elements of finite order is generated by θ_t, of orders q_t ($t = 1, 2, \ldots$). In $\pi_m(Y_{q_t}^{n+1})$ select an element γ_t such that

$$j_{q_t}'\gamma_t = \theta_t, \qquad q_t\gamma_t = 0 \quad \text{if } q_t \neq 2.$$

If $q_t = 2$, $2\gamma_t = i_2 \circ \alpha^n \circ S_* \theta_t$.

Let $d_t = \text{G.C.D.}(p, q_t)$ and $k_t = pq_t/d_t$. Then ${}_p\pi_{m-1}(S^n)$ is generated

by the elements $(q_t/d_t)\theta_t$, generators of cyclic summands; $(q_t/d_t)\theta_t = 0$ if p is prime to q_t. Consider

$$\gamma_t^p = \psi_{p,k_t} \circ \phi_{k_t,q_t} \circ \gamma_t. \tag{7.8}$$

By Lemma 7.6,

$$j'_p(\gamma_t^p) = f_{k_t|p} j'_{q_t}(\gamma_t) = (k_t/p)\theta_t = (q_t/d_t)\theta_t.$$

Therefore, if $d_t > 1$, γ_t^p is of order at least d_t. By Lemma 5.1, even with $d_t = 1$,

$$d_t\gamma_t^p = \tfrac{1}{2}k_t(k_t-1)\mu_{p,q_t} \circ \gamma_t = \tfrac{1}{2}k_t(k_t-1)(q_t/d_t)i_p \circ \alpha^n \circ S_*\theta_t.$$

We deduce from this that γ_t^p is of order d_t unless $p \equiv 2 \pmod 4$ and $q_t = 2$, when

$$2\gamma_t^p = i_p \circ \alpha^n \circ S_*\theta_t$$

and γ_t^p is of order 2 or 4 according as this is zero or not. Our theorem is, for $m \leqslant 2n-2$,

THEOREM 7.9. *$\pi_m(Y_p^{n+1})$ is generated by $i_p \circ \pi_m(S^n)$ and by the non-zero elements γ_t^p (i.e. those with $d_t > 1$). The only relations are those of*

$$i_p \circ \pi_m(S^n),$$

together with

$$\begin{aligned} 2\gamma_t^p &= i_p \circ \alpha^n \circ S_*\theta_t \quad \text{if } q_t = 2,\ p \equiv 2 \pmod 4, \\ d_t\gamma_t^p &= 0 \qquad\qquad\qquad\quad \text{otherwise.} \end{aligned}$$

The multiplication on the left by the generators of $\Pi(Y^{n+1})$ is given by relations (7.10), (7.11):

$$\left.\begin{aligned} 1_p \circ i_p(\theta) &= i_p(\theta) \\ \mu_p \circ i_p(\theta) &= 0 \\ \phi_{pq,p} \circ i_p(\theta) &= q i_{pq}(\theta) \\ \psi_{p,pq} \circ i_{pq}(\theta) &= i_p(\theta) \end{aligned}\right\} \quad (\theta \in \pi_m(S^n)), \tag{7.10}$$

$$\left.\begin{aligned} 1_p \circ \gamma_t^p &= \gamma_t^p \\ \mu_p \circ \gamma_t^p &= (q_t/d_t)i_p \circ \alpha^n \circ S_*\theta_t \\ \phi_{pq,p} \circ \gamma_t^p &= r_t\gamma_t^{pq} + \epsilon i_{pq} \circ \alpha^n \circ S_*\theta_t \\ \psi_{p,pq} \circ \gamma_t^{pq} &= s_t\gamma_t^p + \eta i_p \circ \alpha^n \circ S_*\theta_t \end{aligned}\right\}, \tag{7.11}$$

where $r_t = (pq, q_t)/(p, q_t)$, $s_t = q(p, q_t)/(pq, q_t)$; $\epsilon = 0$ or 1, and $\epsilon = 1$ if and only if $pqq_t/(pq, q_t) \equiv 2 \pmod 4$, and also $r_t \equiv 2$ or $3 \pmod 4$; and $\eta = 0$ or 1, and $\eta = 1$ if and only if $pq_t/(p, q_t) \equiv 2 \pmod 4$ and $s_t \equiv 2$ or $3 \pmod 4$.

These expressions result from calculating the canonical form (as in Lemma 5.2) of

$$\phi_{pq,p} \circ \psi_{p,k_t} \circ \phi_{k_t,q_t}$$

and

$$\psi_{p,pq} \circ \psi_{pq,k_t} \circ \phi_{k_t,q_t},$$

where $k_t = pq_t/(p, q_t)$ and $k'_t = pqq_t/(pq, q_t)$, by using relations (3.4), (3.5), and (3.7).

8. The groups $\{Y_p^{n+1}; S^m\}$ in (Π^{n+1}, S^m)

Let $m \leqslant 2n-3$. Then it follows from § 2, and §§ 5, 9 in (**1**), that there is an exact sequence

$$\pi_{n+1}(S^m) \xrightarrow{i_p^*} \{Y_p^{n+1}; S^m\} \xrightarrow{j_p^*} {}_p\pi_n(S^m) \to 0, \tag{8.1}$$

where†

$$i_p^*(\zeta) = \zeta \circ \{\mu'\}, \qquad i_p^{*-1}(0) = p\pi_{n+1}(S^m), \qquad j_p^*(\xi) = \xi \circ i_p. \tag{8.2}$$

if μ' is as in § 3. The analogue of Lemma 7.6 is

LEMMA 8.3. *If* $\zeta \in \pi_{n+1}(S^m)$, $\xi \in \{Y_{pq}^{n+1}; S^m\}$, $\bar{\xi} \in \{Y_p^{n+1}; S^m\}$, *then*

$$i_{pq}^*(\zeta) \circ \phi_{pq,p} = i_p^*(\zeta), \qquad i_p^*(\zeta) \circ \psi_{p,pq} = i_{pq}^*(S_* f_q \circ \zeta),$$
$$j_p^*(\xi \circ \phi_{pq,p}) = j_{pq}^*(\xi) \circ f_q, \qquad j_{pq}^*(\bar{\xi} \circ \psi_{p,pq}) = j_p^*(\bar{\xi}),$$
$$\bar{\xi} \circ \mu_p = i_p^*(j_p^*(\bar{\xi}) \circ \alpha^n).$$

For we easily see that

$$\{\mu'\} \circ \phi_{pq,p} = \{\mu'\}, \qquad \{\mu'\} \circ \psi_{p,pq} = S_* f_q \circ \{\mu'\},$$
$$\phi_{pq,p} \circ i_p = i_{pq} \circ f_q, \qquad \psi_{p,pq} \circ i_{pq} = i_p.$$

Also $$\bar{\xi} \circ \mu_p = \bar{\xi} \circ i_p \circ \alpha^n \circ \{\mu'\} = i_p^*(j_p^*(\bar{\xi}) \circ \alpha^n).$$

It is shown in (**2**) that, if $\theta \in {}_p\pi_n(S^m)$ is of order q, there is a $\xi \in \{Y_p^{n+1}; S^m\}$ such that

$$j_p^* \xi = \theta, \quad q\xi = \begin{cases} i_p^*(\theta \circ \alpha^n) & \text{if } p \equiv q \equiv 2 \pmod 4, \\ 0 & \text{otherwise.} \end{cases} \tag{8.4}$$

This gives the structure of the groups $\{Y_p^{n+1}; S^m\}$. The analogue of Theorem 7.4 is

THEOREM 8.5. *$\{Y_p^{n+1}; S^m\}$ is an extension of $\pi_{n+1}(S^m)_p$ by ${}_p\pi_n(S^m)$. If $p \neq 2$ is a power of a prime, the extension is the direct sum. If $p = 2$, the extension is determined by the homomorphism*

$$e\colon {}_2\pi_n(S^m) \to i_2^* \pi_{n+1}(S^m)$$

given by $e(\theta) = i_2^*(\theta \circ \alpha^n)$.

This will follow from the previous lemma and the next theorem. The point of considering only powers of primes lies in the analogue of Theorem 7.1, proved in the same way.

Now $\pi_n(S^m)$ is finitely generated, and so can be expressed as a direct sum of cyclic groups whose orders are powers of primes or are infinite;

† No confusion should arise between i_p and i_p^*.

let generators θ_t of orders q_t be chosen for the finite summands in some such decomposition, $t = 1,\ldots$. In $\{Y^{n+1}_{q_t}; S^m\}$ choose ξ_t such that

$$j^*_{q_t}(\xi_t) = \theta_t \quad \text{and} \quad q_t\xi_t = 0 \quad \text{unless } q_t = 2.$$

Let $d_t = (p, q_t)$, and $k_t = pq_t/d_t$; define

$$\xi^p_t = \xi_t \circ \psi_{q_t,k_t} \circ \phi_{k_t,p}. \tag{8.6}$$

Then, as with γ^p_t,

$$j^*_p(\xi^p_t) = (q_t/d_t)\theta_t,$$
$$d_t\xi^p_t = \tfrac{1}{2}k_t(q_t/d_t)(k_t-1)i^*_p(\theta_t \circ \alpha^n),$$

so that $d_t\xi^p_t = 0$ unless $q_t = 2$ and $p \equiv 2 \pmod 4$, when it has the value $i^*_p(\theta_t \circ \alpha^n)$.

Then the analogue of Theorem 7.9 is

THEOREM 8.7. *$\{Y^{n+1}_p; S^m\}$ is generated by $i^*_p\pi_{n+1}(S^m)$ $(\approx \pi_{n+1}(S^m)_p)$ and by the non-zero elements ξ^p_t defined by* (8.5), *with $d_t > 1$. The only relations are those of $i^*_p\pi_{n+1}(S^m)$ together with*

$$2\xi^p_t = i^*_p(\theta_t \circ \alpha^n) \quad \text{if } q_t = 2,\ p \equiv 2 \pmod 4,$$
$$d_t\xi^p_t = 0 \quad \text{otherwise.}$$

The multiplication on the right by the generators of $\Pi(Y^{n+1})$ is given by (8.8), (8.9):

$$\left.\begin{aligned} i^*_p(\zeta) \circ 1_p &= i^*_p(\zeta) \\ i^*_p(\zeta) \circ \mu_p &= 0 \\ i^*_{pq}(\zeta) \circ \phi_{pq,p} &= i^*_p(\zeta) \\ i^*_p(\zeta) \circ \psi_{p,pq} &= qi^*_{pq}(\zeta) \end{aligned}\right\} \quad (\zeta \in \pi_{n+1}(S^m)), \tag{8.8}$$

$$\left.\begin{aligned} \xi^p_t \circ 1_p &= \xi^p_t \\ \xi^p_t \circ \mu_p &= (q_t/d_t)i^*_p(\theta_t \circ \alpha^n) \\ \xi^{pq}_t \circ \phi_{pq,p} &= s_t\xi^p_t + \eta i^*_p(\theta_t \circ \alpha^n) \\ \xi^p_t \circ \psi_{p,pq} &= r_t\xi^{pq}_t + \epsilon i^*_{pq}(\theta_t \circ \alpha^n) \end{aligned}\right\}, \tag{8.9}$$

where r_t, s_t, ϵ, η mean the same as in (7.11). *These last two expressions are obtained by an argument similar to that used in proving* (7.11).

There remains one problem: when $m = n$, $n+1$, there are essential maps $S^m \to Y^{n+1}_p$, $Y^{n+1}_q \to S^m$ whose compositions we must determine. It is obvious that all compositions

$$S^m \to Y^{n+1}_p \to S^m \quad (m = n, n+1)$$

are inessential, and also that all essential maps

$$Y^{n+1}_p \to S^m \to Y^{n+1}_q \quad (m = n, n+1),$$

if any, are multiples of $\mu_{q,p}$.

The results are summarized in

LEMMA 8.10. *If ι^k, α^k respectively generate $\pi_k(S^k)$, $\pi_{k+1}(S^k)$, then*

$$i_q \circ i_p^*(\alpha^n) = \mu_{q,p} = i_q \circ \alpha^n \circ i_p^*(\iota^{n+1}),$$

and all other classes of compositions

$$Y_p^{n+1} \to S^m \to Y_q^{n+1}, \qquad S^m \to Y_p^{n+1} \to S^m \quad (m = n, n+1)$$

are zero.

On the other hand, the general problem of determining the compositions

$$S^m \to Y_p^{n+1} \to S^r,$$

where $m \leqslant 2n-2$, $n \leqslant 2r-3$, is not solved.

Appendix

Theorem 5.1 in (**6**) has a special case:

THEOREM A. *If $f\colon SK_1 \to SK_2$, where K_1, K_2 are* CW*-complexes, K_2 is $(r-2)$-connected, and* $\dim K_1 \leqslant 2r-3$, *then $f \simeq Sg$, for some (cellular) map $g\colon K_1 \to K_2$.*

This can be proved from, and implies, a lemma derived from Theorem II in (**11**):

LEMMA. *If $r \geqslant 3$, and K is $(r-2)$-connected,*

$$S_*\colon \pi_q(K) \to \pi_{q+1}(SK)$$

is an isomorphism for $q \leqslant 2r-4$, and onto for $q \leqslant 2r-3$.

This lemma is used in the proof of J. H. C. Whitehead's theorem (unpublished),

THEOREM B. *If K is an $(r-1)$-connected* CW*-complex and*

$$\dim K \leqslant 2r-1,$$

then there is an $(r-2)$-connected CW*-complex L with* $\dim L = \dim K + 1$, *such that K and SL are of the same homotopy type.*

I have already referred in (**2**) to this result. It is clear that K^r is of the homotopy type of a cluster of r-spheres; let L^{r-1} be a similar cluster of $(r-1)$-spheres so chosen that there is a homotopy equivalence $f_r\colon SL^{r-1} \to K^r$. Suppose that L^{n-1}, f_n have been constructed so that f_n is a homotopy equivalence $f_n\colon SL^{n-1} \to K^n$. By the lemma,

$$f_n \circ S_*\colon \pi_{n-1}(L^{n-1}) \to \pi_n(K^n)$$

is onto for $n-1 \leqslant 2r-3$. Let the $(n+1)$-cells of K^{n+1} be attached to K^n by maps g_i: then there exist maps $g_i'\colon S^{n-1} \to L^{n-1}$ such that $f_n \circ Sg_i'$ is homotopic to g_i. Let L^n be formed by attaching cells with attaching maps g_i': by Lemma 3 of (**10**), we can extend f_n to a homotopy equivalence $f_{n+1}\colon SL^n \to K^{n+1}$. Thus L can be constructed skeleton by skeleton.

Let K be as in Theorem B, with $\dim K \leqslant 2r-2$, $\nu_i\colon SL_i \to K$ $(i = 1, 2)$ classes of homotopy equivalences. Then ν_2 possesses an inverse ν_2^{-1}, and by Theorem A there is a class of maps $\theta\colon L_1 \to L_2$ such that

$$S_*\theta = \nu_2^{-1} \circ \nu_1.$$

We can use this to introduce an additive structure in $\{K; X\}$, for any simply-connected X, in a slightly different way from that of § 3. For ν_1 determines a one-to-one transformation onto,

$$\nu_1\colon \{K; X\} \to \{SL_1; X\}$$

given by $\nu_1(\alpha) = \alpha \circ \nu_1$; since X is simply-connected, the set $\{SL_1; X\}$ can be identified with the elements of the track group $(L_1)^1(X; x_*)$, $x_* \in X$. Also θ determines a homomorphism $(L_2)^1(X; x_*) \to (L_1)^1(X; x_*)$ [§ 2, (**1**)], which is a homomorphism $\{SL_2; X\} \to \{SL_1; X\}$ such that $\alpha \to \alpha \circ S_*\theta$. Therefore, since ν_i are classes of homotopy equivalences, the homomorphism is an isomorphism onto. Define addition in $\{K; X\}$ by the rule

$$\alpha+\beta = (\alpha \circ \nu_1+\beta \circ \nu_1) \circ \nu_1^{-1}.$$

Then the addition is independent of the choices of L_i, ν_i. The addition in our application in § 3 is abelian because (Theorem B) each L is of the homotopy type of a suspension SM, and, if X is 2-connected,

$$(SM)^1(X; x_*) \approx (M)^2(X; x_*).$$

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UNIVALENT REGIONS OF INTEGRAL FUNCTIONS

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1. THE object of this paper is to prove the following:

THEOREM. *To every integral function $f(z)$ of order $0 < \rho < \frac{1}{2}$ corresponds an infinite sequence of simply connected domains D_q whose minimum distance from the origin tends to infinity with q, and a corresponding sequence of numbers A_q such that $\lim A_q = \infty$ $(q \to \infty)$ with the following property: each domain D_q is mapped by $w = f(z)$ univalently either onto $|w| < A_q$ or onto this circle with not more than $2e(2-\log\cos\pi\rho)\sec\pi\rho$ non-intersecting segments joining interior points to the circumference removed.*

Valiron (**2**) has proved the above theorem for $\rho = 0$ with

$$2e(2-\log\cos\pi\rho)\sec\pi\rho$$

replaced by 2 and has conjectured that this result is true for $\rho < \frac{1}{2}$. The method of proof will depend on Wiman's classical result for integral functions of order less than $\frac{1}{2}$, and on the idea of *proximate* orders.

2. The notation used is as follows. $f(z)$ is an integral function of order ρ $(0 < \rho < \frac{1}{2})$. The maximum value of $|f(z)|$ on the circle $|z| = r$ is denoted by $M(r,f)$. The number of zeros of $f(z)$ in $|z| \leqslant r$ is $n(r,f)$; $n(0,f)$ is the order of the zero of $f(z)$ at the origin. The number of zeros of $f(z)$ in $r \leqslant |z| \leqslant R$ is $n(r, R;f)$.

The proof of the theorem requires a series of lemmas.

LEMMA 1. (**1**) *To an integral function $f(z)$ of order ρ there corresponds a continuous function $\rho(r)$ (a proximate order of f) such that*

(i) *$\rho(r)$ is differentiable for $r \geqslant r_0$, except at isolated points at which $\rho'(r-0)$ and $\rho'(r+0)$ exist;*

(ii) $\lim_{r\to\infty} \rho(r) = \rho$;

(iii) $\lim_{r\to\infty} r\rho'(r)\log r = 0$;

(iv) $\limsup_{r\to\infty} \dfrac{\log M(r,f)}{r^{\rho(r)}} = 1.$

Quart. J. Math. Oxford (2), 5 (1954), 291–6.

LEMMA 2. [(3) 128.] *If $\rho(r)$ is a proximate order of $f(z)$, then, for an infinite sequence of r,*

$$\log|f(z)| > (\cos\pi\rho-\epsilon)r^{\rho(r)} \quad (\epsilon > 0).$$

LEMMA 3. *If $\rho(r)$ is a proximate order of $f(z)$, then for all suitably large r*

$$n(r,f) \leqslant e(\rho+\epsilon)r^{\rho(r)} \quad (\epsilon > 0).$$

By Jensen's theorem,

$$\int_0^r \frac{n(x,f)-n(0,f)}{x}\,dx + n(0,f)\log r = \frac{1}{2\pi}\int_0^{2\pi} \log|f(re^{i\theta})|\,d\theta - \log|c|,$$

where c is the coefficient of $z^{n(0,f)}$ in $f(z)$. When r is large enough, then

$$\log M(r,f) < (1+\eta)r^{\rho(r)} \quad (\eta > 0),$$

so that, with $k > 1$,

$$\begin{aligned}\int_0^{kr} \frac{n(x,f)}{x}\,dx &\leqslant \int_0^{kr} \frac{n(x,f)-n(0,f)}{x}\,dx + n(0,f)\log r \\ &= \frac{1}{2\pi}\int_0^{2\pi} \log|f(kre^{i\theta})|\,d\theta - \log|c| \\ &\leqslant \log M(kr,f)-\log|c| < (1+\eta)(kr)^{\rho(kr)},\end{aligned}$$

and thus

$$n(r,f) \leqslant \frac{(1+\eta)(kr)^{\rho(kr)}}{\log k}.$$

Now it is easily shown that $r^{\rho(r)-\rho-\delta}$ is ultimately a decreasing function of r for any $\delta > 0$. Hence, when r is large enough,

$$(kr)^{\rho(kr)-\rho-\delta} \leqslant r^{\rho(r)-\rho-\delta},$$

which gives

$$(kr)^{\rho(kr)} \leqslant k^{\rho+\delta}r^{\rho(r)}.$$

Therefore

$$n(r,f) \leqslant \frac{(1+\eta)k^{\rho+\delta}r^{\rho(r)}}{\log k}.$$

The minimum value of $k^{\rho+\delta}/\log k$ is $e(\rho+\delta)$, so that finally

$$n(r,f) \leqslant e(\rho+\delta)(1+\eta)r^{\rho(r)}.$$

It is apparent that by choosing η and δ sufficiently small the lemma follows.

LEMMA 4. *If $\rho(r)$ is a proximate order of $f(z)$, then*

$$n(r,f) \geqslant \left\{\frac{\rho\cos\pi\rho}{2-\log\cos\pi\rho} - \epsilon\right\}r^{\rho(r)} \quad (\epsilon > 0)$$

for all sufficiently large r for which Lemma 2 is true.

From Jensen's theorem,

$$\int_0^r \frac{n(x,f)-n(0,f)}{x}\,dx + n(0,f)\log r > (\cos\pi\rho-\eta)r^{\rho(r)}-\log|c|$$

for $\eta > 0$, provided that r is a value for which Lemma 2 is true, with ϵ replaced by η. Hence, with $0 < \mu < 1$,

$$\int_{\mu r}^r \frac{n(x,f)}{x}\,dx \geqslant (\cos\pi\rho-2\eta)r^{\rho(r)} - \int_{r_0}^{\mu r} \frac{n(x,f)}{x}\,dx$$

if r is taken large enough. When the result of Lemma 3 is used in the integral on the right-hand side of the above inequality, we get

$$\begin{aligned} n(r,f)\log\frac{1}{\mu} &\geqslant (\cos\pi\rho-2\eta)r^{\rho(r)}-e(\rho+\delta)\int_{r_0}^{\mu r}\frac{x^{\rho(x)}}{x}\,dx \quad (\delta>0) \\ &= (\cos\pi\rho-2\eta)r^{\rho(r)}-e(\rho+\delta)\int_{r_0}^{\mu r} x^{\rho(x)-\rho+\delta}x^{\rho-\delta-1}\,dx \\ &\geqslant (\cos\pi\rho-2\eta)r^{\rho(r)}-\frac{e(\rho+\delta)}{\rho-\delta}\mu^{\rho-\delta}r^{\rho(r)} \end{aligned}$$

since $r^{\rho(r)-\rho+\delta}$ is ultimately an increasing function of r for any $\delta > 0$. Thus

$$\begin{aligned} n(r,f) &\geqslant \frac{[\cos\pi\rho-2\eta-\{e(\rho+\delta)/(\rho-\delta)\}\mu^{\rho-\delta}]}{\log(\mu^{-1})}r^{\rho(r)} \\ &\geqslant \left\{\frac{\cos\pi\rho-e\mu^{\rho}-\gamma}{\log(\mu^{-1})}\right\}r^{\rho(r)} \end{aligned}$$

when η, δ are suitably chosen. It can be shown that

$$\frac{\cos\pi\rho-e\mu^{\rho}-\gamma}{\log(\mu^{-1})} \geqslant \frac{\rho(\cos\pi\rho-\gamma)}{2-\log(\cos\pi\rho-\gamma)},$$

and so the result follows when γ is taken small enough.

LEMMA 5. *If r is so large that Lemma 2 is true, then*

(i) $n(r,f) \geqslant \left\{\dfrac{\rho\cos\pi\rho}{2-\log\cos\pi\rho}-\epsilon\right\}r^{\rho(r)}$,

(ii) $n(r,f') \leqslant e(\rho+\epsilon)r^{\rho(r)}$,

where $\rho(r)$ is a proximate order of $f(z)$.

Lemma 5 will follow from Lemmas 3 and 4 if it can be shown that a proximate order of $f(z)$ is also a proximate order of $f'(z)$. This will be the case if

$$\limsup_{r\to\infty}\frac{\log M(r,f')}{r^{\rho(r)}} = 1.$$

Now $$f(z) = \int_0^z f'(\zeta)\, d\zeta + f(0),$$

which gives $$M(r,f) \leqslant rM(r,f') + |f(0)|.$$

Therefore $$\frac{\log M(r,f)}{r^{\rho(r)}} \leqslant \frac{\log M(r,f')}{r^{\rho(r)}} + \frac{\log r}{r^{\rho(r)}} + \frac{K}{\log r}$$

for some positive constant K. When the limit is taken of both sides, then

$$1 = \limsup_{r\to\infty} \frac{\log M(r,f)}{r^{\rho(r)}} \leqslant \limsup_{r\to\infty} \frac{\log M(r,f')}{r^{\rho(r)}}.$$

By Cauchy's theorem,

$$f'(z) = \frac{1}{2\pi i} \oint \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

Choosing the contour of integration to be $|\zeta - z| = \delta$ gives

$$M(r,f') \leqslant \frac{1}{\delta} M(r+\delta, f),$$

and so

$$\limsup_{r\to\infty} \frac{\log M(r,f')}{r^{\rho(r)}} \leqslant \limsup_{r\to\infty} \frac{\log M(r+\delta,f)}{r^{\rho(r)}}$$

$$\leqslant \limsup_{r\to\infty} \frac{(r+\delta)^{\rho(r+\delta)}}{r^{\rho(r)}}.$$

Since $r^{\rho(r)-\rho-\eta}$ $(\eta > 0)$ is ultimately a decreasing function of r, it follows that, if r is large enough,

$$(r+\delta)^{\rho(r+\delta)-\rho-\eta} \leqslant r^{\rho(r)-\rho-\eta},$$

and thus $$\frac{(r+\delta)^{\rho(r+\delta)}}{r^{\rho(r)}} \leqslant \left(\frac{r+\delta}{r}\right)^{\rho+\eta},$$

which shows that

$$\limsup_{r\to\infty} \frac{\log M(r,f')}{r^{\rho(r)}} \leqslant \limsup_{r\to\infty} \left(\frac{r+\delta}{r}\right)^{\rho+\eta} = 1.$$

Hence $$\limsup_{r\to\infty} \frac{\log M(r,f')}{r^{\rho(r)}} = 1,$$

and $\rho(r)$ is also a proximate order of $f'(z)$.

LEMMA 6. *It is possible to choose a sequence of annuli such that on the boundaries of each member of the sequence Lemma 2 is true and also*

(i) $n(r, R; f) \geqslant \left\{\frac{\rho \cos \pi\rho}{2 - \log\cos \pi\rho} - \epsilon\right\} R^{\rho(R)}$,

(ii) $n(r, R; f') \leqslant e(\rho+\epsilon) R^{\rho(R)}$.

Suppose that Lemma 2 is true for r, R. Since $n(r,f) \to \infty$, it follows that, given r, we can choose R so that $n(r,f)/n(R,f) < \eta$ for any $\eta > 0$. Thus

$$\begin{aligned} n(r, R;f) &= n(R,f) - n(r,f) \\ &\geqslant n(R,f)(1-\eta) \\ &\geqslant \left\{ \frac{\rho \cos \pi\rho}{2 - \log\cos \pi\rho} - \epsilon \right\} R^{\rho(R)}, \end{aligned}$$

when η is taken small enough. Also

$$\begin{aligned} n(r, R;f') &= n(R,f') - n(r,f') \\ &\leqslant n(R,f')' \\ &\leqslant e(\rho+\epsilon) R^{\rho(R)}, \end{aligned}$$

whence the lemma follows.

LEMMA 7. *If Lemma 2 is true on the boundaries of the annulus (r, R) and Lemma 6 holds in this annulus, then the set of points in (r, R) such that*

$$|f(z)| < (\cos \pi\rho - \epsilon) r^{\rho(r)}, \qquad \cos \pi\rho > \epsilon > 0,$$

denoted by E, is open and is the union of a finite non-zero number of simply-connected components. Each component contains at least one zero of $f(z)$ and is mapped p times onto $|w| < (\cos \pi\rho - \epsilon) r^{\rho(r)}$ by $w = f(z)$, where p is the number of zeros in the component.

By Lemma 6 the annulus contains at least one zero of $f(z)$ and therefore E is non-empty. If E contains all points on a simple closed curve, then it follows from the maximum-modulus principle that all points within the curve also belong to E. This shows that each component of the set is simply connected. If z_0 belongs to E, then

$$|f(z_0)| < (\cos \pi\rho - \epsilon) r^{\rho(r)} \leqslant (\cos \pi\rho - \epsilon)(1-\eta) r^{\rho(r)}$$

for some $\eta > 0$. From the continuity of $f(z)$ it follows that for some $\delta(\eta)$ all points of the circle $|z - z_0| < \delta$ satisfy

$$|f(z)| < (\cos \pi\rho - \epsilon)(1 - \tfrac{1}{2}\eta) r^{\rho(r)} < (\cos \pi\rho - \epsilon) r^{\rho(r)},$$

i.e. the set E is open.

By Rouché's theorem, in each component $f(z) - w$, $|w| < (\cos \pi\rho - \epsilon) r^{\rho(r)}$ has the same number of zeros as $f(z)$. Since $f(z)$ is non-constant, it takes at least one value once, and so must contain at least one zero. Therefore the number of components of E does not exceed the number of zeros of $f(z)$ in the annulus. The final point of the lemma follows from the above.

3. We are now in a position to prove the theorem. Let $\{(r_i, R_i)\}_1^\infty$ be a sequence of annuli such that on the boundaries of each annulus Lemma 2

holds and such that the zeros of $f(z)$ and $f'(z)$ in each annulus satisfy Lemma 6. There is at least one component of the set of points in the annulus (r_i, R_i) satisfying

$$|f(z)| < (\cos \pi\rho - \epsilon) r_i^{\rho(r_i)}$$

which contains α zeros of $f(z)$ and not more than

$$\alpha e(2 - \log\cos \pi\rho)\sec \pi\rho + \epsilon$$

zeros of $f'(z)$. If this were not the case, then Lemma 6 would be contradicted.

When z describes the above component, then w describes a Riemann surface of α circles of radius $(\cos \pi\rho - \epsilon) r_i^{\rho(r_i)}$. In one sheet of this surface there must not be more than

$$2e(2 - \log\cos \pi\rho)\sec \pi\rho + \epsilon$$

branch points. The inverse image of this sheet (after the removal of non-intersecting segments joining the branch points to the circumference) is also simply connected. The domain D_i is defined to be the component described above and A_i is defined to be $(\cos \pi\rho - \epsilon) r_i^{\rho(r_i)}$.

It is to be noted that, when $\rho \to 0$, then

$$2e(2 - \log\cos \pi\rho)\sec \pi\rho \to 4e,$$

so that the limiting case disagrees with Valiron's result.

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EVEN QUADRATIC FORMS WITH DETERMINANT UNITY

By F. VAN DER BLIJ (*Utrecht*)

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In this note I give some elementary proofs of theorems concerning positive-definite even quadratic forms with determinant unity.

1. Definitions

Let k denote a positive integer. We define the following functions of τ ($q = e^{\pi i \tau}$, $|q| < 1$):

$$A_k = \tfrac{1}{2}(\vartheta_{00}^{8k} + \vartheta_{01}^{8k} + \vartheta_{10}^{8k}),$$

where $\vartheta_{00} = \sum_n q^{n^2}$, $\vartheta_{01} = \sum_n (-1)^n q^{n^2}$, $\vartheta_{10} = \sum_n q^{(n+\frac{1}{2})^2}$, $\sum_n$ denotes $\sum_{n=-\infty}^{\infty}$

and

$$B_k = \sum_{(m_i)} q^{Q(m_1,\ldots,m_{8k})},$$

where $Q(m_1,\ldots,m_{8k})$ is an even quadratic form with determinant unity. (A quadratic form $a_{11}x_1^2 + 2a_{12}x_1x_2 + \ldots$ is called *even* if the numbers $\frac{1}{2}a_{11}, a_{12}, \ldots$ are rational integers.)

2. Theorems

[1] $A_1^2 = A_2$.

[2] *There exist forms Q_k^* such that the corresponding function B_k^* equals A_k.*

[3] *The $[\frac{1}{3}k]+1$ functions $A_1^{k-3x}A_3^x$, with a fixed integer k, are linearly independent. The function A_k can be represented as a sum of these functions with constant coefficients.*

3. Modular forms

If we use the theory of modular forms, parts of the theorems of § 2 become trivial facts. From the well-known transformation formulae of theta functions it can be deduced that A_k is a modular form of dimension $-4k$. Using the transformation theory of multiple theta functions (**2**), (**5**) it can be proved that B_k is a modular form of dimension $-4k$. We know there exist $[\frac{1}{3}k]+1$ linearly independent modular forms of dimension $-k$. Thus $A_1 = B_1$ and $A_1^2 = A_2 = B_2$.

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4. Proofs of the theorems

[1] I first prove $\vartheta_{00}^4 = \vartheta_{01}^4 + \vartheta_{10}^4$, a well-known formula from the theory of theta functions. I give an elementary proof, which illustrates the methods used in the proof of [2]. Write

$$\vartheta_{10}^k = \sum_{N=1}^{\infty} \rho_k(N) q^N,$$

$$\vartheta_{00}^k = 1 + \sum_{N=1}^{\infty} r_k(N) q^N,$$

$$\vartheta_{01}^k = 1 + \sum_{N=1}^{\infty} (-1)^N r_k(N) q^N.$$

We have $\rho_4(2N) = 0$ since $(n+\frac{1}{2})^2 \equiv \frac{1}{4} \pmod 2$. I show that, if N is an odd integer, $\rho_4(N) = 2r_4(N)$. Let odd integers u, v, w, s satisfy

$$u^2+v^2+w^2+s^2 = 4N.$$

Then there are as many solutions with $\sum u \equiv 0 \pmod 4$ as with $\sum u \equiv 2 \pmod 4$. For, if $\sum u \equiv 0$ or $2 \pmod 4$, then

$$-u+v+w+s \equiv 2 \text{ or } 0 \pmod 4.$$

Let x, y, z, t be integers satisfying

$$x^2+y^2+z^2+t^2 = N.$$

Then, when $\sum u \equiv 0 \pmod 4$, we have a one-to-one correspondence between the two sets of variables given by

$$u = x+y+z+t, \qquad v = x+y-z-t,$$
$$w = x-y+z-t, \qquad s = x-y-z+t,$$

since this gives

$$\sum u^2 = 4 \sum x^2, \qquad 4x = u+v+w+s$$

and u, v, w, s are odd integers.

We proved $$r_4(N) = (-1)^N r_4(N) + \rho_4(N):$$

that is to say $\vartheta_{00}^4 = \vartheta_{01}^4 + \vartheta_{10}^4$.

From this we find after a simple calculation

$$A_1^2 - A_2 = \tfrac{1}{4}(\vartheta_{00}^8 + \vartheta_{01}^8 + \vartheta_{10}^8)^2 - \tfrac{1}{2}(\vartheta_{00}^{16} + \vartheta_{01}^{16} + \vartheta_{10}^{16}) = 0.$$

[2] $A_k = 1 + \sum_{N=1}^{\infty} \lambda_k(N) q^N,$

where $$\lambda_k(N) = \tfrac{1}{2}\{r_{8k}(N) + (-1)^N r_{8k}(N) + \rho_{8k}(N)\}.$$

It may be readily seen that $\lambda_k(N) = 0$ if N equals an odd integer.

If N is an even integer, we have

$$\lambda_k(N) = r_{8k}(N) + \tfrac{1}{2}\rho_{8k}(N).$$

Thus $\lambda_k(2N)$ equals the number of the integral solutions of $\sum_{i=1}^{8k} y_i^2 = 8N$ with

$$y_1 \equiv y_2 \equiv \ldots \equiv y_{8k} \pmod 2, \qquad y_1+y_2+\ldots+y_{8k} \equiv 0 \pmod 4,$$

since every 'even' representation (y) satisfies the second congruence and half the number of the 'odd' representations (y) satisfy the second congruence. Now there is a one-to-one correspondence between the systems (y) satisfying the congruence relations and all systems (z) of integers established by

$$\left.\begin{aligned} y_1 &= 4z_1-2z_2-\ldots-2z_{8k-1}-z_{8k} \\ y_j &= \qquad 2z_j \qquad\qquad\quad -z_{8k} \\ y_{8k} &= \qquad\qquad\qquad\quad -z_{8k} \end{aligned}\right\} \quad (j = 2, 3, \ldots, 8k-1).$$

We find that $\lambda_k(2N)$ denotes the number of integral solutions of

$$4Q^*(z_1,\ldots,z_{8k}) = (4z_1-2z_2\ldots)^2 + \sum_{j=2}^{8k-1} (2z_j-z_{8k})^2 + z_{8k}^2 = 8N.$$

It can be readily seen that $Q^*(z_1,\ldots,z_{8k})$ is an even quadratic form. The determinant of the substitution $y \to z$ equals -2^{8k}. Thus

$$|4Q^*| = 2^{16k}, \qquad |Q^*| = 1.$$

Hence we have proved

$$A_k = 1 + \sum \lambda_k(N) q^N = 1 + \sum \lambda_k(2N) q^{2N} = \sum_{(m_i)} q^{Q^*(m_1,\ldots,m_{8k})}.$$

[3] We prove that A_k can be represented by a linear combination of functions $A_1^{k-3\alpha}A_3^{\alpha}$. This is trivial if $k = 1, 2, 3$. We proceed by mathematical induction. We use the identity

$$A_k = 2A_1A_{k-1} - A_1^2A_{k-2} + \tfrac{2}{3}(A_3 - A_1^3)A_{k-3} \quad (k \geqslant 4). \tag{1}$$

In order to prove this identity we note that

$$(x-\vartheta_{00}^8)(x-\vartheta_{01}^8)(x-\vartheta_{10}^8) \equiv x^3 + \alpha_1 x^2 + \alpha_2 x + \alpha_3,$$

where

$$\alpha_1 = -2A_1, \qquad \alpha_2 = 2A_1^2 - A_2 = A_1^2,$$

$$\alpha_3 = -\tfrac{2}{3}(A_3 + 2A_1^3 - 3A_1A_2) = \tfrac{2}{3}(A_1^3 - A_3).$$

Thus

$$x^k = 2A_1x^{k-1} - A_1^2x^{k-2} + \tfrac{2}{3}(A_3 - A_1^3)x^{k-3}$$

has roots ϑ_{00}^8, ϑ_{01}^8, and ϑ_{10}^8. By substituting these roots and adding the equalities thus obtained, we can prove the formula (1).

Since $\vartheta_{00}^8\vartheta_{01}^8\vartheta_{10}^8 \neq 0$, we have $A_3 \neq A_1^3$.

Now we prove that the functions $A_1^{k-3\alpha}A_3^{\alpha}$ are linearly independent. For let

$$\sum \mu_\alpha A_1^{k-3\alpha}A_3^{\alpha} = 0.$$

Then $\sum \mu_\alpha (A_3A_1^{-3})^\alpha = 0$; thus $A_3A_1^{-3}$ equals a constant number. Since $A_3 = 1+\ldots$ and $A_1 = 1+\ldots$, we find $A_3A_1^{-3} = 1$. Since $A_3 \neq A_1^3$, we have proved that the functions $A_1^{k-3\alpha}A_3^{\alpha}$ are linearly independent.

5. Conclusion

Since there is only one class of even quadratic forms with determinant unity in 8 variables (**3**), we have $A_1 = B_1$. There are two classes of such forms in 16 variables, but we have for every form Q_2 the identity $A_2 = B_2$ (**6**). Using the theory of modular forms, we have proved that every modular form of dimension $-4k$ can be represented by a sum of multiple theta series.

Using the simple proof given by B. van der Pol (**4**) of the formulae

$$A_1 = 1+240\sum_{N=1}^{\infty}\sigma_3(N)q^{2N},$$

$$A_2 = 1+480\sum_{N=1}^{\infty}\sigma_7(N)q^{2N},$$

we have found a simple proof of the fact that the number of the representations of an even integer $2N$ by an even quadratic form with determinant unity in 8 and 16 variables is given by $240\sigma_3(N)$ and $480\sigma_7(N)$ respectively. (**1**), (**5**).

We remark that

$$A_3 - A_1^3 = 3\times 2^7\Delta_3,$$

where

$$\Delta_3 = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{N=1}^{\infty}\tau(N)q^N.$$

Here $\tau(N)$ denotes the number-theory function of S. Ramanujan. If we denote the Eisenstein series of dimension -12 by

$$C_3 = 1+\tfrac{65520}{691}\sum_{N=1}^{\infty}\sigma_{11}(N)q^{2N},$$

we find in due course

$$691A_1^3 = 432{\cdot}000\Delta_3+691C_3,$$

$$691A_3 = 697{\cdot}344\Delta_3+691C_3.$$

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A NEW EXPANSION FOR THE MODIFIED BESSEL FUNCTION $K_0(z)$

By C. P. SINGER (*Hayes*)

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THE following is a new expansion for $K_0(z)$ in terms of $I_{2n}(z)$, where the singularity at the origin is isolated into a single logarithmic factor to the first term only, while the remaining terms of the series are regular everywhere. The expansion is

$$K_0(z) = -(\gamma+\log \tfrac{1}{2}z)I_0(z)+2\sum_{n=1}^{\infty}\frac{1}{n}I_{2n}(z),$$

where γ is Euler's constant.

Proof. We have

$$K_0(z) = -(\gamma+\log \tfrac{1}{2}z)I_0(z)+\sum_{m=1}^{\infty}\frac{(\frac{1}{2}z)^{2m}}{(m!)^2}\sum_{k=1}^{m}\frac{1}{k},$$

and so the theorem asserts that

$$\sum_{n=1}^{\infty}\frac{1}{n}I_{2n}(z) = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(\frac{1}{2}z)^{2n}}{(n!)^2}\sum_{k=1}^{n}\frac{1}{k},$$

i.e., since

$$I_{2n}(z) = \sum_{m=0}^{\infty}\frac{(\frac{1}{2}z)^{2(n+m)}}{m!\,(2n+m)!},$$

that

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{m=0}^{\infty}\frac{(\frac{1}{2}z)^{2(n+m)}}{m!\,(2n+m)!} = \frac{1}{2}\sum_{m=1}^{\infty}\frac{(\frac{1}{2}z)^{2m}}{(m!)^2}\sum_{k=1}^{m}\frac{1}{k}.$$

The coefficient of $(\frac{1}{2}z)^{2n}$ on the left is

$$\sum_{r=1}^{n}\frac{1}{r(n-r)!\,(n+r)!}$$

and on the right is

$$\frac{1}{2(n!)^2}\sum_{k=1}^{n}\frac{1}{k}.$$

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Thus the theorem requires the identity

$$\sum_{r=1}^{n} \frac{(n!)^2}{r(n-r)!\,(n+r)!} = \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k}.$$

To prove this result, let the left-hand member be $\frac{1}{2}S_n$; then

$$S_n = 2\sum_{r=1}^{n} \frac{r^{-1}(n!)^2}{(n+r)!\,(n-r)!},$$

$$\begin{aligned}
\Delta S_n &= S_n - S_{n-1} \\
&= 2\{(n-1)!\}^2 \sum_{r=1}^{n} \frac{r^{-1}\{n^2-(n^2-r^2)\}}{(n+r)!\,(n-r)!} \\
&= \{(n-1)!\}^2 \sum_{r=1}^{n} \frac{(n+r)-(n-r)}{(n+r)!\,(n-r)!} \\
&= \{(n-1)!\}^2 \sum_{r=1}^{n} \left\{\frac{1}{(n+r-1)!\,(n-r)!} - \frac{1}{(n+r)!\,(n-r-1)!}\right\} \\
&= \frac{\{(n-1)!\}^2}{n!\,(n-1)!} = \frac{1}{n}
\end{aligned}$$

since there is zig-zag cancelling in $\sum$.

Thus $$S_n = \sum(\Delta S_n) = \sum_{k=1}^{n} \frac{1}{k},$$

and so $$\sum_{r=1}^{n} \frac{(n!)^2}{r(n+r)!\,(n-r)!} = \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k}.$$

This result can easily be generalized. For, using the Wronskian, we know (see, for instance, N. W. McLachlan, *Bessel Functions*, p. 116, Ex. 20) that

$$I_\nu(z)K_{\nu+1}(z) + I_{\nu+1}(z)K_\nu(z) = z^{-1},$$

where ν, z are in general complex. When ν is a positive integer n, we thus obtain the following set of equations:

$$\frac{K_{n+1}}{I_{n+1}} = \frac{1}{zI_{n+1}I_n} - \frac{K_n}{I_n}, \qquad -\frac{K_n}{I_n} = -\frac{1}{zI_nI_{n-1}} + \frac{K_{n-1}}{I_{n-1}},$$

. . . . . .

$$(-1)^n \frac{K_1}{I_1} = (-1)^n \frac{1}{zI_1I_0} - (-1)^n \frac{K_0}{I_0}.$$

Adding all the equations, we have

$$K_{n+1}(z) = (-1)^{n+1} I_{n+1}(z) \left\{ \frac{K_0(z)}{I_0(z)} + \frac{1}{z} \sum_{r=1}^{n+1} \frac{(-1)^r}{I_r(z) I_{r-1}(z)} \right\},$$

where n is a positive integer or zero. When n is a negative integer, we have $K_\nu(z) = K_{-\nu}(z)$ and so $K_{-n}(z) = K_n(z)$.

These expansions are convenient in permitting operations on K_n to be carried out in terms of I_n (a function which is regular everywhere and tabulated in British Association tables) with the exception of one term only which has a singularity at the origin.

EMBEDDING IN HOMOGENEOUS SPACES

By M. SHIMRAT (*Jerusalem*)

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1. A TOPOLOGICAL (or metric) space S is said to be (metrically) *homogeneous* if, for any two points a, b of S, there exists an (isometric) homeomorphism of S onto itself carrying a into b.

A completely regular space can be embedded in a (normal) homogeneous space. This follows from the fact that it can be embedded in a cube [see (**3**) 29] and therefore in a product of 1-spheres. It is the main object of this note to prove that any space can be embedded in a homogeneous space and that the separation and connectivity properties of the given space can be preserved in the process of embedding.

2. THEOREM 1. *Any topological space S can be embedded in a homogeneous space S^* so that*

(i) *if S is T_0, T_1, Hausdorff, regular, normal, completely normal* [*see* (**1**)] *or perfectly normal,*† *the same is true of S^*;*

(ii) *if S is T_1, then S is closed in S^*; and*

(iii) *if S is connected or locally connected, so is S^*.*

Before carrying out the actual construction of the embedding space I shall explain the intuitive geometrical idea behind this construction.

A homogeneous space may be thought of as one in which every point is in the same relation to the rest of the space. Let S be the given space which is (perhaps) not homogeneous. We provide ourselves with an unlimited supply of copies of S (in each of which we denote the points by the same letters as the corresponding points of S). Let a and b be any two distinct points of S. We take a copy of S, which we call S_{ab}, and attach it, at its point b, to the point a of S. We do this for every ordered pair of distinct points a, b.

We now treat each S_{ab} in the same manner, by attaching fresh copies of S at each point, except at b; and we go on like this indefinitely.

In order to see that the space S^* thus constructed is homogeneous, consider first the configuration S_0 obtained as follows. Take a collection

† A space is *perfectly normal* if, for any two disjoint closed sets A, B in S, there exists a continuous real function on S with $f(A) = 0$, $f(B) = 1$ and $0 < f(p) < 1$ elsewhere.

$\{S_{(c)}\}$ of disjoint S-spaces (i.e. copies of S), one for each point c of S. In each $S_{(c)}$ take the point c and identify all these 'diagonal' points into a single point.

Returning now to S^*, if a_0 is a fixed point of S, then the various S-spaces passing through a_0 are the $S_{a_0 b}$ $(b \neq a_0)$ and S itself, which may be thought of as $S_{a_0 a_0}$. We therefore have a bundle of S-spaces through a_0, homeomorphic to the above S_0. To every point other than a_0 of every one of these S-spaces there are attached new S-spaces, and so on. This picture is the same for every point of S^*, which is therefore homogeneous.

3. We are now going to prove the theorem. Let S be the given space. We construct the embedding space S^* as follows.

Consider the group G generated by all points a of S with the defining relations

$$a^2 = 1 \quad (a \in S).$$

Every element g of G can be expressed uniquely by a normal word (empty for the unit 1)

$$g \equiv a_1 a_2 \ldots a_n \quad (n \geqslant 0),$$

i.e. one in which $a_i \neq a_{i+1}$ throughout. The sign $\equiv$ is meant to imply that the right-hand side is a normal word. I shall call the a_i the *coordinates* of g.

Those elements g for which n is even form a subgroup E of G of index 2. The coset $G - E$ consists of all elements of G with odd n. We now put $S^* = G - E$ and we shall presently introduce a topology in S^* by defining neighbourhoods of points.

(The connexion with the intuitive description given in § 2 is as follows. The space S_{ab} of § 2 is the set abS in the present notation. The point b of S_{ab} is attached to the point a of S because $abb = a$.)

I shall call a *neighbourhood field* (or a *field*) any correspondence N assigning to each element

$$d \equiv a_1 a_2 \ldots a_{2k-1} \quad (k \geqslant 1)$$

of S^* a neighbourhood $N(d)$ in S of its last coordinate. Thus, for a given N, $N(d)$ is determined by d and is a neighbourhood of a_{2k-1} in S. For the sake of convenience we shall assume that in S the neighbourhoods of a point are the open sets containing it.

For every field N we denote by K_N the set of all points

$$e \equiv a_1 a_2 \ldots a_{2l} \quad (l \geqslant 0) \tag{3.1}$$

of E such that

$$a_{2i} \in N(a_1 a_2 \ldots a_{2i-1}) \quad (i = 1, 2, \ldots, l). \tag{3.2}$$

We now define as neighbourhoods in S^* of a point p of S^* all the sets pK_N, where N is any field.

4. In order to prove that S^* has now been made into a topological space we have to show that the neighbourhood axioms are satisfied by the sets pK_N.

(i) Obviously any point has neighbourhoods. Also $p \in pK_N$ since $1 \in K_N$ for every N; $l = 0$ in (3.1) and (3.2).

(ii) If pK_{N_1}, pK_{N_2} are neighbourhoods of p, we form the field N_3 defined by: $N_3(d) = N_1(d) \cap N_2(d)$ for every $d \in S^*$. Clearly

$$pK_{N_3} = pK_{N_1} \cap pK_{N_2}.$$

(iii) Given a neighbourhood pK_N and a point $q \in pK_N$, we have to show that some neighbourhood $qK_{N'}$ is contained in pK_N.

Let $q = pe$ where

$$e \in K_N, \qquad e \equiv a_1 a_2 \ldots a_{2m} \quad (m \geqslant 0). \tag{4.1}$$

We define a field N' by

$$N'(d) = N(ed) \quad (d \in S^*).$$

We have to show first that $N'(d)$ is a neighbourhood of the last coordinate of d. This is clear in case the last coordinate of ed is that of d. The only other case is when all the coordinates of d cancel out in the product ed, i.e. when

$$d \equiv a_{2m} a_{2m-1} \ldots a_{2j} \quad (1 \leqslant j \leqslant m).$$

We then have

$$ed \equiv a_1 a_2 \ldots a_{2j-1},$$

and $N(ed)$ is a neighbourhood of a_{2j-1}. By (4.1) we have

$$a_{2j} \in N(a_1 a_2 \ldots a_{2j-1}) = N(ed).$$

As an open set containing a_{2j}, the set $N(ed)$ is a neighbourhood of a_{2j}, i.e. of the last coordinate of d.

We shall now prove that $qK_{N'} \subseteq pK_N$. Let qe' be any point of $qK_{N'}$, i.e.

$$e' \equiv a'_1 a'_2 \ldots a'_{2s} \quad (s \geqslant 0),$$

where

$$a'_{2i} \in N'(a'_1 a'_2 \ldots a'_{2i-1}) \quad (i = 1, 2, \ldots, s).$$

We have $qe' = pee'$ and we want to prove that $ee' \in K_N$.

In the product $ee' = a_1 a_2 \ldots a_{2m} a'_1 a'_2 \ldots a'_{2s}$ a certain number t ($\geqslant 0$) of coordinates at the end of e cancel out with t coordinates at the beginning of e'. We therefore have

$$ee' \equiv a_1 a_2 \ldots a_{2m-t} a'_{t+1} a'_{t+2} \ldots a'_{2s}$$

(where the a_i's and/or the a'_i's may be missing). On the right-hand side we have, for every a_{2i} $(2 \leqslant 2i \leqslant 2m-t)$,

$$a_{2i} \in N(a_1 a_2 \ldots a_{2i-1})$$

since $e \in K_N$, and for every a'_{2i} $(t+1 \leqslant 2i \leqslant 2s)$, which is an even-numbered coordinate of ee',

$$\begin{aligned} a'_{2i} \in N'(a'_1 a'_2 \ldots a'_{2i-1}) &= N(e a'_1 a'_2 \ldots a'_{2i-1}) \\ &= N(a_1 a_2 \ldots a_{2m-t} a'_{t+1} a'_{t+2} \ldots a'_{2i-1}). \end{aligned}$$

We therefore have $ee' \in K_N$, as required.

5. S^* has been shown to be a topological space and we shall now see that S is embedded in S^*. Since $S \subseteq S^*$, we have to show that the relative topology of S as a point set in S^* coincides with that originally given in S. For this it suffices to prove that for any $a \in S$ the neighbourhoods of a in S are exactly all the sets $aK_N \cap S$, where N is any field. This follows from the fact that the point ae $(a \in S, e \in K_N)$ lies in S if and only if either $e = 1$ or $e \equiv aa'$ $(a' \in N(a) - \{a\})$, and therefore $aK_N \cap S = N(a)$.

6. *S^* is a homogeneous space.* In fact, for any fixed $e \in E$, the mapping $p \to ep$ of S^* onto itself is an automorphism of S^* since it carries a neighbourhood pK_N into the neighbourhood epK_N and vice versa.

If now p_0, q_0 are given points in S^*, the mapping $p \to q_0 p_0^{-1} p$ is an automorphism of S^* carrying p_0 into q_0. This proves the homogeneity of S^*.

7. Suppose now that S is T_0 or T_1. We shall prove that S^* then has the same property.

Let us assume first that S is T_0 and let p, q be any two distinct points of S^*. We put

$$e = p^{-1}q \equiv a_1 a_2 \ldots a_{2l} \quad (l > 0)$$

and consider the two distinct points a_1, a_2 of S. We have at least one of the following cases:

(i) There exists a neighbourhood U_1 of a_1 in S, with $a_2 \notin U_1$. Then we take a field N_1 such that $N_1(a_1) = U_1$ and we have

$$q = pe = pa_1 a_2 \ldots a_{2l} \notin pK_{N_1}.$$

(ii) There exists a neighbourhood U_2 of a_2 in S with $a_1 \notin U_2$. Then we take a field N_2 such that $N_2(a_{2l} a_{2l-1} \ldots a_2) = U_2$ and we have

$$p = qe^{-1} = qa_{2l} a_{2l-1} \ldots a_1 \notin qK_{N_2}.$$

Thus S^* is T_0. By considering case (i) only we can see that, if S is T_1, so is S^*.

We shall also show here that, if S is T_1, it is closed in S^*. This means that any point q of $S^* - S$ has a neighbourhood disjoint from S. We have

$$q \equiv aa_1a_2...a_{2l} \quad (l > 0).$$

If we determine N_2 as in (ii), then, as may easily be verified, $S \cap qK_{N_2} = 0$, which proves our assertion.

8. Let now S be Hausdorff. We shall prove that S^* is then Hausdorff.

Let p, q, e and the a_i be as in § 7. We take a field N such that

$$N(a_1a_2...a_{2l-1}) \cap N(a_{2l}) = 0 \tag{8.1}$$

and show that we then have $pK_N \cap qK_N = 0$.

Indeed, let qe' be any point of qK_N and put

$$e' \equiv a'_1a'_2...a'_{2m} \quad (m \geqslant 0).$$

We shall show that $qe' \notin pK_N$. We have

$$qe' = pee' = p(a_1a_2...a_{2l})(a'_1a'_2...a'_{2m}).$$

Here two cases may arise:

(i) $$ee' \equiv a_1a_2...a_{2l}a'_1a'_2...a'_{2m}.$$

Then we have, by (8.1), $a_{2l} \notin N(a_1a_2...a_{2l-1})$ and therefore $qe' \notin pK_N$.

(ii) $a'_1 = a_{2l}$. Then

$$a'_2 \in N(a'_1) = N(a_{2l}). \tag{8.2}$$

By (8.1) this implies $a'_2 \neq a_{2l-1}$ and so

$$ee' \equiv a_1a_2...a_{2l-1}a'_2a'_3...a'_{2m}.$$

Here, by (8.1), (8.2), $a'_2 \notin N(a_1a_2...a_{2l-1})$, which again gives $qe' \notin K_N$. Thus S^* is Hausdorff.

9. We next take S to be a regular space and prove that S^* is then regular.

Let p be any point of S^* and pK_N any neighbourhood of p. We have to find a neighbourhood $pK_{N'}$ of p such that its closure $\overline{pK_{N'}}$ is contained in pK_N.

We take a field N' such that

$$\overline{N'(d)} \subseteq N(d) \tag{9.1}$$

for every d of S^*. In order to prove $\overline{pK_{N'}} \subseteq pK_N$ we have to show that any point outside pK_N has a neighbourhood disjoint from $pK_{N'}$. Let therefore $q \notin pK_N$. Then we have $q = pe$, where e is such that in its normal expression some a_{2i} does not satisfy the relation

$$a_{2i} \in N(a_1a_2...a_{2i-1}).$$

We put (as contraction of the normal expression)

$$e \equiv e'abe'' \quad (e', e'' \in \boldsymbol{E};\ a, b \in S),$$

where $$b \notin N(e'a).$$

By (9.1), the point b has a neighbourhood U_b in S such that

$$U_b \cap N'(e'a) = 0. \tag{9.2}$$

Take a field N_1 such that $N_1(e''^{-1}b) = U_b$. (Note that b is the last coordinate of $e''^{-1}b$.) I shall show that qK_{N_1} contains no point of $pK_{N'}$.

Let qe_1 be any point of qK_{N_1}. We have

$$qe_1 = pee_1 = pe'abe''e_1.$$

If b does not cancel out in the product ee_1, then the point qe_1 cannot lie in $pK_{N'}$ as $b \notin N(e'a)$ and *a fortiori* $b \notin N'(e'a)$. We therefore have to consider only the case

$$e_1 \equiv e''^{-1}ba'e_1' \quad (a' \in S,\ e_1' \in E) \tag{9.3}$$

(contracted as above) and

$$qe_1 = pe'abe''.e''^{-1}ba'e_1' = pe'aa'e_1'. \tag{9.4}$$

Since $qe_1 \in qK_{N_1}$, we have, in (9.3), $a' \in N_1(e''^{-1}b) = U_b$. Therefore, by (9.2), $a' \notin N'(e'a)$. This applied to (9.4) shows that $qe_1 \notin pK_{N'}$. We have thus proved that $qK_{N_1} \cap pK_{N'} = 0$, as required.

10. Let now S be normal. We shall prove that S^* is normal.

Let A, B be any closed disjoint sets in S^*. We shall define a continuous real function $f(p)$ on S^* such that

$$f(p) = 0 \text{ on } A, \quad f(p) = 1 \text{ on } B, \quad 0 \leqslant f(p) \leqslant 1 \text{ elsewhere.} \tag{10.1}$$

Suppose that f has been defined and satisfies (10.1) for all points with less than $2k$ coordinates ($k \geqslant 0$). For each $e \equiv a_1 a_2 \ldots a_{2k}$ we define f on the normal set eS so that it is continuous, satisfies (10.1), and (if $k > 0$) agrees with f, as previously defined, at the point ea_{2k}. It is easily seen that the last condition can be fulfilled. (Here we have to make use of the fact that 'normal', in our sense, includes 'T_1'.)

The function f thus defined by induction on the whole of S^* satisfies (10.1) and is single-valued, for the various sets eS above can have only points with less than $2k$ coordinates in common. To prove its continuity we argue as follows. If $f(p)$ lies in the open real interval I, so does $f(pK_N)$ for some neighbourhood pK_N. We fix p and choose N so that, if $d \equiv a_1 a_2 \ldots a_{2l} a_{2l+1}$ ($l \geqslant 0$) and $f(pda_{2l+1}) \in I$, then $f(pdN(d)) \subseteq I$. This can be done since $f(pda)$ is continuous in a ($a \in S$). It is now easily verified that $f(pK_N) \subseteq I$.

11. Let S be completely normal. In order to prove that S^* is completely normal it is sufficient to show that any open set G in S^* is normal. The proof is the same as in § 10, except that

(i) A, B are now disjoint subsets of G, closed relative to G;

(ii) f is defined on G and has to satisfy (10.1) there;

(iii) the field N is subjected to the additional condition: if $pda_{2l+1} \in G$, then $pdN(d) \subseteq G$.

On the other hand, if we substitute $0 < f(p) < 1$ for $0 \leqslant f(p) \leqslant 1$ in (10.1), we can prove exactly as in § 10 that, if S is perfectly normal, so is S^*.

12. We now prove that, if S is connected or locally connected, so is S^*.

Let us take the locally connected case first and let p be any point of S^* and pK_N any neighbourhood of p. We can determine the field N' so that $N'(d) \subseteq N(d)$ throughout and every $N'(d)$ is connected. Then $pK_{N'}$ is a connected neighbourhood of p contained in pK_N. For, if $a_1 \in S$, the set $pa_1 N'(a_1)$ is a connected subset (since it is homeomorphic to $N'(a_1)$) of $pK_{N'}$ and contains p; it therefore lies in that component C_p of $pK_{N'}$ which contains p. Next, if $a_1 a_2 a_3 \equiv a_1 a_2 a_3$ and $a_2 \in N(a_1)$, the set $pa_1 a_2 a_3 N'(a_1 a_2 a_3)$ is a connected subset of $pK_{N'}$ and contains $pa_1 a_2$; it therefore lies in that component of $pK_{N'}$ which contains $pa_1 a_2$, i.e. in C_p. Continuing in this way we can show that the whole of $pK_{N'}$ lies in C_p: in other words $pK_{N'} = C_p$ and is connected.

By taking every $N(d) = N'(d) = S$ in the above we prove that, if S is connected, so is S^*.

This completes the proof of Theorem 1.

13. THEOREM 2. *Any metric space S can be isometrically embedded in a metrically homogeneous space S', so that S is closed in S'.*

(Without the latter condition the theorem follows from the known fact that any metric space may be isometrically embedded in a Banach space [see (**2**) 316]).

We define G, $\equiv$, and E as in § 3 and consider again $S' = G - E$. (The topology of S', however, will not be that of S^*.) We introduce a metric ρ' in S' by

$$\rho'(p, q) = \rho(a_1, a_2) + \rho(a_3, a_4) + \ldots + \rho(a_{2k-1}, a_{2k}),$$

where

$$p^{-1}q \equiv a_1 a_2 \ldots a_{2k} \quad (k \geqslant 0)$$

and ρ is the metric given in S.

In order to show that the triangle axiom holds for ρ', consider a third point r and let

$$q^{-1}r \equiv b_1 b_2 \ldots b_{2l} \quad (l \geqslant 0).$$

Then

$$p^{-1}r = p^{-1}q.q^{-1}r = a_1a_2...a_{2k}b_1b_2...b_{2l}$$
$$\equiv a_1a_2...a_{2k-s}b_{s+1}b_{s+2}...b_{2l} \quad (s \geqslant 0).$$

If s is even, we obviously have

$$\rho'(p,r) \leqslant \rho'(p,q)+\rho'(q,r). \tag{13.1}$$

Otherwise we have $a_{2k-s+1} = b_s$ and therefore

$$\rho(a_{2k-s}, b_{s+1}) \leqslant \rho(a_{2k-s}, a_{2k-s+1})+\rho(b_s, b_{s+1}),$$

and (13.1) follows again.

The other axioms are obviously true for ρ'.

14. S' has been shown to be a metric space. Clearly S is isometrically embedded in S', for, if a, b are distinct points of S, we have

$$a^{-1}b = ab \equiv ab,$$

and so

$$\rho'(a,b) = \rho(a,b).$$

Also S is closed in S', for, if

$$p \equiv a_1a_2...a_{2l-1} \quad (l > 0)$$

is any point of $S'-S$, then p is at the positive distance $\rho(p, a_1)$ from S, as can easily be verified.

15. S' is metrically homogeneous since, for any $e \in E$,

$$(ep)^{-1}.eq = p^{-1}e^{-1}eq = p^{-1}q,$$
$$\rho'(ep, eq) = \rho'(p,q),$$

which proves that the mapping $p \to ep$ is isometric.

This completes the proof of Theorem 2.

It may be noted that the space S' is not necessarily locally connected when S is. A counter-example is given by the case where S is the segment $0 < t < 1$ with the points $\frac{1}{2}, \frac{1}{3}, \frac{1}{4},...$ removed and with the usual metric. On the other hand it is easily shown, as for S^*, that, if S is connected, so is S'.

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RIEMANN EXTENSIONS OF AFFINE CONNECTED SPACES

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1. Introduction

RIEMANN extensions have been defined by Patterson and Walker (**4**), who have shown how a Riemannian structure can be given to the $2n$-dimensional tangent bundle of an n-dimensional manifold with given non-Riemannian structure. By this means it is possible to relate the properties of a non-Riemannian n-space with those of certain Riemannian $2n$-spaces. An example of this, given by Patterson (**2**), led to the definition of a harmonic affine connected space. Further examples will be given in the present paper, particularly in relation to symmetric and recurrent spaces.

We shall be concerned mainly with local properties, although it will become evident that some of our results hold in the large. We shall also assume that the space and tensor fields are of class C^∞ or C^ω.

2. Riemann extensions of order r

A Riemannian space R^{2n} of $2n$ dimensions will be called *a Riemann extension* if it admits a parallel null n-plane and if, for some integer $r \geqslant 1$, the vectors† $\lambda^i_{(\alpha)}$ of any given basis of the n-plane satisfy the relations

$$R_{hijp_1p_2\ldots p_r}\lambda^h_{(\epsilon)}\lambda^{p_1}_{(\alpha_1)}\lambda^{p_2}_{(\alpha_2)}\ldots\lambda^{p_r}_{(\alpha_r)} = 0 \quad \begin{pmatrix} i, j = 1, 2, \ldots, 2n \\ \epsilon, \alpha_1, \ldots, \alpha_r = 1, 2, \ldots, n \end{pmatrix}, \tag{1}$$

where R_{hijk} is the curvature tensor of R^{2n} and the comma denotes covariant differentiation with respect to the Christoffel symbols of R^{2n}. If r is the least integer for which these relations are satisfied, we shall say that the extension is of *order* r.

It follows at once that:

(i) a flat Riemannian $2n$-space is a Riemann extension of the first order since $R_{hijk} = 0$;

(ii) a symmetric Riemannian $2n$-space which admits a parallel field of null n-planes is a Riemann extension of the first or the second order since $R_{hijk,l} = 0$.

† In the present paper, Latin suffixes take values 1, 2,..., $2n$, unprimed Greek suffixes take values 1, 2,..., n, and primed Greek suffixes take values $n+1$,..., $2n$ with the convention that $\alpha' = \alpha + n$.

It will now be shown that, for a Riemann extension of order r, a coordinate system (x^α, ξ_α) $(\xi_\alpha = x^{\alpha+n})$ exists in terms of which the metric takes the form

$$\left.\begin{aligned}(ds)^2 &= g_{\alpha\beta}\,dx^\alpha dx^\beta + 2\,dx^\alpha d\xi_\alpha,\\ \text{where}\qquad g_{\alpha\beta} &= c_{\alpha\beta} - 2\Gamma^{\rho_1}_{\alpha\beta}\xi_{\rho_1} + T^{\rho_1\rho_2}_{\alpha\beta}\xi_{\rho_1}\xi_{\rho_2} + \ldots + T^{\rho_1\rho_2\ldots\rho_r}_{\alpha\beta}\xi_{\rho_1}\xi_{\rho_2}\ldots\xi_{\rho_r}\end{aligned}\right\}. \tag{2}$$

Here $c_{\alpha\beta}$, $\Gamma^{\rho_1}_{\alpha\beta}$, and $T^{\rho_1\rho_2\ldots\rho_\nu}_{\alpha\beta}$ $(\nu = 2, 3, \ldots, r)$ are independent of the ξ's and are symmetric in α and β. Conversely R^{2n} with metric of this form is a Riemann extension of order r provided that $T^{\rho_1\rho_2\ldots\rho_r}_{\alpha\beta}$ are not all zero. Thus (2) is a canonical form for the metric of a Riemann extension of order r.

To derive this form we use the fact [Walker (7)] that, for a Riemannian $2n$-space R^{2n} admitting a parallel field of null n-planes, there is a coordinate system (x^α, ξ_α) in terms of which the metric of R^{2n} takes the form

$$(ds)^2 = g_{\alpha\beta}(x, \xi)\,dx^\alpha dx^\beta + 2\,dx^\alpha d\xi_\alpha, \tag{3}$$

a basis for the null plane being given by $\delta^i_{(\alpha')}$ $(\alpha' = n+1, \ldots, 2n)$. For this metric, the Christoffel symbols

$$\begin{Bmatrix}\alpha\\ i\ \beta'\end{Bmatrix}, \quad \begin{Bmatrix}i\\ \alpha'\ \beta'\end{Bmatrix}$$

are zero, and it follows that

$$R_{\alpha'\beta'\gamma\nu'_1} = 0, \quad R_{\alpha'\beta'\gamma\nu'_1,\nu'_2} = 0, \quad R_{\alpha'\beta'\gamma\nu'_1,\nu'_2\nu'_3} = 0, \quad \ldots.$$

Hence
$$R_{\alpha'\beta\gamma\nu'_1} = \tfrac{1}{2}g_{\beta\gamma,\alpha'\nu'_1},$$

and, for any s,
$$R_{\alpha'\beta\gamma\nu'_1,\nu'_2\ldots\nu'_s} = \tfrac{1}{2}g_{\beta\gamma,\alpha'\nu'_1\nu'_2\ldots\nu'_s}, \tag{4}$$

where the full point denotes partial differentiation. Substituting in (1) and remembering that the basis for the null n-plane is $\delta^i_{(\alpha')}$, we see that

$$g_{\beta\gamma,\alpha'\nu'_1\nu'_2\ldots\nu'_r} = 0.$$

Hence $g_{\alpha\beta}$ is a polynomial of at most degree r in the ξ's and the metric (3) is of the form (2).

Conversely, consider the R^{2n} given by the metric (2). We have from Walker's canonical form that the space with metric (2) admits a parallel null n-plane with basis vectors $\delta^i_{(\alpha')}$. This is an extension of order r if

$$R_{\alpha'\beta\gamma\nu'_1,\nu'_2\ldots\nu'_s} \neq 0, \qquad R_{\alpha'\beta\gamma\nu'_1,\nu'_2\ldots\nu'_r} = 0 \quad (s < r).$$

This follows from (4) since $g_{\alpha\beta}$ in (2) is a polynomial of degree r in the ξ's.

The coefficients $c_{\alpha\beta}$, $\Gamma^\gamma_{\alpha\beta}$, and $T^{\rho_1\rho_2\ldots\rho_s}_{\alpha\beta}$ $(1 < s \leqslant r)$ in the expression for $g_{\alpha\beta}$ in (2) may be any functions of the coordinates x^α. These coordinates can be taken to define a point of an n-dimensional space M^n on which

the $c_{\alpha\beta}$, $\Gamma^{\gamma}_{\alpha\beta}$, and $T^{\rho_1\rho_2\cdots\rho_s}_{\alpha\beta}$ $(1 < s \leqslant r)$ then define a structure of some kind. For M^n to be defined uniquely by R^{2n}, it is necessary for this structure to be preserved by all the transformations which leave invariant a prescribed form for the metric of the Riemann extension. This prescribed form will be a particular case of the general canonical form (2).

The most general transformation that leaves the metric (2) formally invariant is given by

$$\bar{x}^{\alpha} = f^{\alpha}(x), \qquad \bar{\xi}_{\alpha} = \frac{\partial x^{\beta}}{\partial \bar{x}^{\alpha}}\{\xi_{\beta} + \eta_{\beta}(x)\}, \tag{5}$$

the f's being functions with non-vanishing Jacobian. The effect of this transformation on the metric is to replace $g_{\alpha\beta}$ by $\bar{g}_{\alpha\beta}$, where

$$\bar{g}_{\alpha\beta} = \left(g_{\gamma\delta} - \frac{\partial \eta_{\gamma}}{\partial x^{\delta}} - \frac{\partial \eta_{\delta}}{\partial x^{\gamma}}\right)\frac{\partial x^{\gamma}}{\partial \bar{x}^{\alpha}}\frac{\partial x^{\delta}}{\partial \bar{x}^{\beta}} - 2\frac{\partial^2 x^{\gamma}}{\partial \bar{x}^{\alpha}\partial \bar{x}^{\beta}}(\xi_{\gamma} + \eta_{\gamma}). \tag{6}$$

The transformation (5) may be restricted so that (6) gives some desired transformation of a structure defined on M^n. For example, taking $\eta_{\beta} = 0$ in (5), the coefficients $\Gamma^{\gamma}_{\alpha\beta}$ are transformed as connexion coefficients while $c_{\alpha\beta}$ and $T^{\nu_1\nu_2\cdots\nu_s}_{\alpha\beta}$ $(1 < s \leqslant r)$ are transformed as tensor components. In this case, therefore, there is defined in M^n a symmetric affine connexion $\Gamma^{\gamma}_{\alpha\beta}$ together with the tensors $c_{\alpha\beta}$, $T^{\nu_1\nu_2\cdots\nu_s}_{\alpha\beta}$ $(1 < s \leqslant r)$.

When $r = 2$, the quantities $c_{\alpha\beta}$ and $T^{\nu_1\nu_2}_{\alpha\beta}$ may be chosen so that $\Gamma^{\gamma}_{\alpha\beta}$ are transformed as connexion coefficients of a conformal Riemannian n-space [Patterson and Walker (4)].

The case $r = 1$ is of special interest since $\Gamma^{\gamma}_{\alpha\beta}$ transform as connexion coefficients under the transformation (5) and thus define a symmetric affine connexion in M^n.

The remainder of the present paper will be concerned with this case. By 'a Riemann extension of an affine connected space A^n' we mean a Riemann extension R^{2n} given by the canonical form

$$\left.\begin{aligned} (ds)^2 &= g_{\alpha\beta}\,dx^{\alpha}dx^{\beta} + 2\,dx^{\alpha}d\xi_{\alpha} \\ g_{\alpha\beta} &= c_{\alpha\beta} - 2\Gamma^{\rho}_{\alpha\beta}\xi_{\rho} \end{aligned}\right\}, \tag{7}$$

where $c_{\alpha\beta} = c_{\beta\alpha}$ are any functions of the coordinates x^{α} and $\Gamma^{\gamma}_{\alpha\beta} = \Gamma^{\gamma}_{\beta\alpha}$ are the connexion coefficients of A^n. The extension will be described as *restricted* when the $c_{\alpha\beta}$ are taken to be zero.

3. The curvature tensor of R^{2n}

From (7), the independent components of the Christoffel symbols which do not vanish identically are

$$\begin{Bmatrix} \alpha \\ \beta\ \gamma \end{Bmatrix} = -\begin{Bmatrix} \beta' \\ \alpha'\ \gamma \end{Bmatrix} = \Gamma^{\alpha}_{\beta\gamma}, \qquad \begin{Bmatrix} \alpha' \\ \beta\ \gamma \end{Bmatrix} = \tfrac{1}{2}(g_{\alpha\beta,\gamma} + g_{\alpha\gamma,\beta} - g_{\beta\gamma,\alpha} - 2g_{\alpha\rho}\Gamma^{\rho}_{\beta\gamma}).$$

It follows by direct calculation that, if $B^{\alpha}_{\beta\gamma\delta}$ are the components of the curvature tensor of A^n, the independent components of the curvature tensor R_{hijk} of R^{2n} which are not identically zero are given by

$$R_{\alpha\beta\gamma\delta} = B_{\beta\gamma\delta,\alpha}, \tag{8}$$

$$R_{\alpha\beta\gamma\delta} = \tfrac{1}{2}S_{\beta\alpha\delta\gamma} + \xi_\rho(B^\rho_{\alpha\delta\gamma;\beta} + B^\rho_{\beta\gamma\delta;\alpha}) + \\ + \tfrac{1}{2}(g_{\mu\alpha}B^\mu_{\beta\gamma\delta} + g_{\mu\beta}B^\mu_{\alpha\delta\gamma} + g_{\mu\gamma}B^\mu_{\delta\alpha\beta} + g_{\mu\delta}B^\mu_{\gamma\beta\alpha}), \tag{9}$$

where

$$S_{\beta\alpha\delta\gamma} = c_{\alpha\delta;\gamma\beta} + c_{\beta\gamma;\delta\alpha} - c_{\alpha\gamma;\delta\beta} - c_{\beta\delta;\gamma\alpha} - c_{\rho\alpha}B^\rho_{\beta\gamma\delta} - c_{\rho\beta}B^\rho_{\alpha\delta\gamma} \tag{10}$$

and the semi-colon denotes covariant differentiation with respect to $\Gamma^\alpha_{\beta\gamma}$.

If R^{2n} is flat, then $R_{hijk} = 0$, and it follows that $B^{\alpha}_{\beta\gamma\delta} = 0$, i.e. that A^n is flat. Also the c's satisfy

$$S_{\alpha\beta\gamma\delta} = 0,$$

which must clearly be the conditions, in the case of a flat A^n, that the c's can be 'transformed away' by a transformation of the form (5). Conversely, R^{2n} is flat when A^n is flat and $c_{\alpha\beta}$ satisfy $S_{\alpha\beta\gamma\delta} = 0$.

In the following section we shall seek conditions under which $c_{\alpha\beta}$ can be transformed away without assuming that A^n is flat.

4. Transformation of $c_{\alpha\beta}$

When $\eta_\alpha = 0$ in (5), the $c_{\alpha\beta}$ are transformed as tensor components in A^n. If, therefore, the $c_{\alpha\beta}$ are to be transformed away, it is sufficient to consider a transformation of the form

$$\bar{x}^\alpha = x^\alpha, \qquad \bar{\xi}_\alpha = \xi_\alpha + \eta_\alpha(x). \tag{11}$$

This leaves $\Gamma^\alpha_{\beta\gamma}$ unaltered but $c_{\alpha\beta}$ is replaced by $\bar{c}_{\alpha\beta}$, where

$$\bar{c}_{\alpha\beta} = c_{\alpha\beta} - \eta_{\alpha;\beta} - \eta_{\beta;\alpha}.$$

Consequently $c_{\alpha\beta}$ can be transformed away if we can find η_α to satisfy the equations

$$\eta_{\alpha;\beta} + \eta_{\beta;\alpha} = c_{\alpha\beta}. \tag{12}$$

Writing

$$\phi_{\alpha\beta} = \eta_{\alpha;\beta} - \eta_{\beta;\alpha},$$

and using the Ricci identities

$$\eta_{\alpha;\beta\gamma} - \eta_{\alpha;\gamma\beta} = \eta_\rho B^\rho_{\alpha\beta\gamma},$$

we find that the system of equations (12) is equivalent to the mixed system of partial differential equations for η_α, $\phi_{\alpha\beta}$ given by

$$\left.\begin{aligned} \eta_{\alpha;\beta} &= \tfrac{1}{2}(c_{\alpha\beta} + \phi_{\alpha\beta}) && (13)\\ \phi_{\beta\gamma;\alpha} &= c_{\alpha\beta;\gamma} - c_{\gamma\alpha;\beta} - 2\eta_\rho B^\rho_{\alpha\beta\gamma} && (14)\\ \phi_{\alpha\beta} + \phi_{\beta\alpha} &= 0 && (15) \end{aligned}\right\} \text{(P)}$$

The conditions of integrability arising from (13) are satisfied identically because of (14) and (15). Those arising from (14) are found to be

$$2\eta_\rho D^\rho_{\delta\alpha\beta\gamma}+\phi_{\rho\mu}E^{\rho\mu}_{\delta\alpha\beta\gamma}+S_{\delta\alpha\beta\gamma}=0, \tag{16}$$

where $S_{\delta\alpha\beta\gamma}$ is given by (10),

$$D^\rho_{\delta\alpha\beta\gamma}=B^\rho_{\delta\beta\gamma;\alpha}-B^\rho_{\alpha\beta\gamma;\delta},$$

and

$$E^{\rho\mu}_{\delta\alpha\beta\gamma}=\delta^\mu_\alpha B^\rho_{\delta\beta\gamma}+\delta^\mu_\beta B^\rho_{\gamma\alpha\delta}-\delta^\mu_\gamma B^\rho_{\beta\alpha\delta}-\delta^\mu_\delta B^\rho_{\alpha\beta\gamma}.$$

Further conditions arise when we differentiate (16) covariantly and substitute from (13) and (14).

A case of particular interest arises when the system (P) is completely integrable, i.e. when conditions (16) are satisfied identically as a consequence of (15). The conditions for this are

$$D^\rho_{\delta\alpha\beta\gamma}=0, \qquad S_{\delta\alpha\beta\gamma}=0, \tag{17}$$

and

$$E^{\rho\mu}_{\delta\alpha\beta\gamma}=E^{\mu\rho}_{\delta\alpha\beta\gamma}. \tag{18}$$

Contracting μ and δ in (18), we get

$$B^\rho_{\alpha\beta\gamma}=\frac{1}{n-1}(\delta^\rho_\gamma B_{\beta\alpha}-\delta^\rho_\beta B_{\gamma\alpha}-2\delta^\rho_\alpha \beta_{\beta\gamma}), \tag{19}$$

where $B_{\alpha\beta}=b_{\alpha\beta}+\beta_{\alpha\beta}$ is the contracted curvature tensor $B^\rho_{\alpha\beta\rho}$ of A^n and $b_{\alpha\beta}$, $\beta_{\alpha\beta}$ are its symmetric and anti-symmetric parts. Contracting ρ and α in (19) and using the fact that $B^\rho_{\rho\alpha\beta}=2\beta_{\beta\alpha}$, we get $\beta_{\alpha\beta}=0$, i.e.

$$B_{\alpha\beta}=B_{\beta\alpha}.$$

Hence (19) becomes

$$B^\rho_{\alpha\beta\gamma}+\frac{1}{n-1}(\delta^\rho_\beta B_{\gamma\alpha}-\delta^\rho_\gamma B_{\alpha\beta})=0. \tag{20}$$

In the case of a space A^n in which $\beta_{\alpha\beta}=0$, the left-hand side of (20) is the component $W^\rho_{\alpha\beta\gamma}$ of the Weyl tensor. It is known [(**1**) § 34] that, if this tensor vanishes, then A^n ($n>2$) is projectively flat. If $n=2$, the conditions for A^n to be projectively flat are $B_{\alpha\beta;\gamma}=B_{\alpha\gamma;\beta}$, and these are satisfied because of the first equation in (17). For $n\geqslant 2$, therefore, A^n is projectively flat. Also, as in the present case, if $\beta_{\alpha\beta}=0$ a coordinate system (x^α) can be chosen so that the connexion coefficients of A^n take the form

$$\Gamma^\alpha_{\beta\gamma}=\delta^\alpha_\beta\psi_\gamma+\delta^\alpha_\gamma\psi_\beta, \qquad \psi_\alpha=\frac{\partial\psi}{\partial x^\alpha} \tag{21}$$

for some function $\psi(x)$.

It can be verified that the curvature tensor of the affine space given by the connexion (21) satisfies the equations $D^\rho_{\delta\alpha\beta\gamma}=0$ and also equations (18).

To see what the conditions $S_{\delta\alpha\beta\gamma} = 0$ mean geometrically, consider the conformal tensor C_{hijk} of R^{2n} given by

$$C_{hijk} = R_{hijk} + \frac{1}{2(n-1)}(R_{hj}g_{ik} + R_{ik}g_{hj} - R_{hk}g_{ij} - R_{ij}g_{hk}) + \\ + \frac{R}{2(n-1)(2n-1)}(g_{hk}g_{ij} - g_{hj}g_{ik}),$$

where R_{ij} are the components of the Ricci tensor of R^{2n} and R its scalar curvature. From (7), (8), and (9)

$$R_{\alpha'\beta} = 0 = R_{\alpha'\beta'}, \qquad R_{\alpha\beta} = B_{\alpha\beta} + B_{\beta\alpha} = 2b_{\alpha\beta}, \qquad R = 0. \quad (22)$$

Thus the independent components of C_{hijk} which do not vanish identically are

$$C_{\alpha'\beta\gamma\delta} = F^{\alpha}_{\beta\gamma\delta}, \quad (23)$$

$$C_{\alpha\beta\gamma\delta} = \xi_\rho D^{\rho}_{\beta\alpha\gamma\delta} + \tfrac{1}{2}S_{\beta\alpha\delta\gamma} + \\ + \tfrac{1}{2}(g_{\mu\alpha}F^{\mu}_{\beta\gamma\delta} + g_{\mu\beta}F^{\mu}_{\alpha\delta\gamma} + g_{\mu\gamma}F^{\mu}_{\delta\alpha\beta} + g_{\mu\delta}F^{\mu}_{\gamma\beta\alpha}), \quad (24)$$

where $S_{\beta\alpha\delta\gamma}$ is given by (10),

$$D^{\rho}_{\beta\alpha\gamma\delta} = B^{\rho}_{\beta\gamma\delta;\alpha} - B^{\rho}_{\alpha\gamma\delta;\beta},$$

$$F^{\alpha}_{\beta\gamma\delta} = B^{\alpha}_{\beta\gamma\delta} + \frac{1}{n-1}(\delta^{\alpha}_{\gamma}b_{\beta\delta} - \delta^{\alpha}_{\delta}b_{\beta\gamma}).$$

Now, if R^{2n} is conformally flat, $C_{hijk} = 0$ and, from (23), $F^{\alpha}_{\beta\gamma\delta} = 0$. Contract α and β; then $\beta_{\gamma\delta} = 0$, so that $F^{\alpha}_{\beta\gamma\delta}$ is the Weyl tensor of A^n. It can also be shown that $b_{\alpha\beta;\gamma} = b_{\alpha\gamma;\beta}$. Thus, for $n \geqslant 2$, A^n is projectively flat with connexion coefficients of the form (21) in a suitable coordinate-system. It follows at once from (24) that $S_{\beta\alpha\gamma\delta} = 0$, and hence the conditions of integrability of the system (P) are satisfied identically; i.e. the $c_{\alpha\beta}$ can be transformed away. Conversely, we have that, if A^n is projectively flat and if the $c_{\alpha\beta}$ can be transformed away, then R^{2n} is conformal to a flat space. We thus have the following†

THEOREM 1. *If the Riemann extension of A^n is conformally flat, then $c_{\alpha\beta}$ can be transformed away and A^n is projectively flat with connexion of the form*

$$\Gamma^{\alpha}_{\beta\gamma} = \delta^{\alpha}_{\beta}\psi_{\gamma} + \delta^{\alpha}_{\gamma}\psi_{\beta}, \qquad \psi_{\alpha} = \frac{\partial\psi}{\partial x^{\alpha}}$$

in some suitable coordinate system and for some function $\psi(x)$. These conditions are sufficient.

† This theorem was given by Walker (8).

Under the conditions of this theorem there exists a coordinate system (x^α, ξ_α) in terms of which the metric of R^{2n} takes the form

$$(ds)^2 = (-2\Gamma^\rho_{\alpha\beta}\,\xi_\rho)\,dx^\alpha dx^\beta + 2\,dx^\alpha d\xi_\alpha,$$

where $$\Gamma^\rho_{\alpha\beta} = \delta^\rho_\alpha \psi_\beta + \delta^\rho_\beta \psi_\alpha, \qquad \psi_\alpha = \frac{\partial \psi(x)}{\partial x^\alpha}.$$

Putting $\zeta_\alpha = e^{-2\psi}\xi_\alpha$, we get

$$(ds)^2 = 2e^{2\psi}\,dx^\alpha d\zeta_\alpha.$$

This exhibits the metric of R^{2n} in a conformally flat form.

5. Cartan-symmetric extensions

Consider the Riemann extension R^{2n} given by the metric (7) and let $R_{hijk,l}$ be the covariant derivative of the curvature tensor R_{hijk}. It can be verified that the independent components of $R_{hijk,l}$ which do not vanish identically are given by

$$R_{\alpha'\beta\gamma\delta,\rho} = B^\alpha_{\beta\gamma\delta;\rho}, \tag{25}$$

$$R_{\alpha\beta\gamma\delta,\rho'} = D^\rho_{\beta\alpha\gamma\delta}, \tag{26}$$

$$\begin{aligned} R_{\alpha\beta\gamma\delta,\rho} = {} & \tfrac{1}{2} S_{\beta\alpha\delta\gamma;\rho} + \xi_\mu (D^\mu_{\beta\alpha\gamma\delta;\rho} - \Gamma^\mu_{\nu\rho} D^\nu_{\beta\alpha\gamma\delta} - \mathscr{B}^\mu_{\rho\beta\alpha\gamma\delta}) + \\ & + \tfrac{1}{2}(g_{\mu\alpha} B^\mu_{\beta\gamma\delta;\rho} + g_{\mu\beta} B^\mu_{\alpha\delta\gamma;\rho} + g_{\mu\gamma} B^\mu_{\delta\alpha\beta;\rho} + g_{\mu\delta} B^\mu_{\gamma\beta\alpha;\rho}) - \\ & - \tfrac{1}{2}(c_{\rho\mu\alpha} B^\mu_{\beta\gamma\delta} + c_{\rho\mu\beta} B^\mu_{\alpha\delta\gamma} + c_{\rho\mu\gamma} B^\mu_{\delta\alpha\beta} + c_{\rho\mu\delta} B^\mu_{\gamma\beta\alpha}), \end{aligned} \tag{27}$$

where $$D^\rho_{\beta\alpha\gamma\delta} = B^\rho_{\beta\gamma\delta;\alpha} + B^\rho_{\alpha\delta\gamma;\beta}, \qquad c_{\rho\mu\alpha} = c_{\rho\mu;\alpha} - c_{\rho\alpha;\mu}$$

and

$$\begin{aligned} \mathscr{B}^\mu_{\rho\beta\alpha\gamma\delta} &= B^\mu_{\rho\nu\alpha} B^\nu_{\beta\gamma\delta} + B^\mu_{\rho\nu\beta} B^\nu_{\alpha\delta\gamma} + B^\mu_{\rho\nu\gamma} B^\nu_{\delta\alpha\beta} + B^\mu_{\rho\nu\delta} B^\nu_{\gamma\beta\alpha} \\ &= B^\mu_{\rho\beta\alpha;\gamma\delta} - B^\mu_{\rho\beta\alpha;\delta\gamma} + B^\mu_{\rho\gamma\delta;\beta\alpha} - B^\mu_{\rho\gamma\delta;\alpha\beta} \end{aligned} \tag{28}$$

because of the Ricci identities.

Suppose that R^{2n} is Cartan-symmetric. Then $R_{hijk,l} = 0$, and it follows from (25) that $B^\alpha_{\beta\gamma\delta;\rho} = 0$, i.e. A^n is symmetric. Hence

$$D^\rho_{\beta\alpha\gamma\delta} = 0, \qquad \mathscr{B}^\mu_{\rho\beta\alpha\gamma\delta} = 0.$$

Substituting in (27), we get

$$S_{\beta\alpha\delta\gamma;\rho} - (c_{\rho\mu\alpha} B^\mu_{\beta\gamma\delta} + c_{\rho\mu\beta} B^\mu_{\alpha\delta\gamma} + c_{\rho\mu\gamma} B^\mu_{\delta\alpha\beta} + c_{\rho\mu\delta} B^\mu_{\gamma\beta\alpha}) = 0. \tag{29}$$

It is interesting to note that, in the case of a symmetric A^n, equation (29) is the next set of conditions of integrability of the system (P) in § 4, obtained by differentiating (16) covariantly and substituting from (13) and (14). Consequently, if equations (16) are algebraically compatible, then $c_{\alpha\beta}$ can be transformed away. However, equations (16) are not necessarily compatible. To illustrate this, consider $\Gamma^\alpha_{\beta\gamma} = 0$ and $c_{\alpha\beta} = x^\alpha x^\beta$ in (7), giving a symmetric Riemann extension of a flat A^n.

In this case equations (16) are not compatible and the $c_{\alpha\beta}$ cannot therefore be transformed away.

When $c_{\alpha\beta} = 0$ in (7), conditions (29) are identically satisfied and we have

THEOREM 2. *If the Riemann extension of A^n is symmetric, then A^n is symmetric. Conversely, if A^n is symmetric, the restricted extension of A^n is symmetric.*

6. Recurrent Riemann extensions

Suppose now that R^{2n} is recurrent [Ruse (**5**)], so that

$$R_{hijk,l} = R_{hijk}\kappa_l$$

for some non-zero vector κ_l called the *vector of recurrence*. It is known that [Walker (**6**)] κ_l is a gradient of some function $\kappa(x, \xi)$. From (8) and (25),

$$B^{\alpha}_{\beta\gamma\delta;\rho} = B^{\alpha}_{\beta\gamma\delta}\kappa_\rho, \tag{30}$$

and hence A^n is recurrent.† From (30) we deduce that κ_ρ is a function of the first n coordinates only; and, since $R_{\alpha'\beta\gamma\delta,\rho'} = 0$, it follows that $\kappa_{\rho'} = 0$, i.e. κ is a function of the x^α only. Hence, from (26),

$$B^{\rho}_{\beta\gamma\delta;\alpha} + B^{\rho}_{\alpha\delta\gamma;\beta} = 0. \tag{31}$$

Conversely let A^n be recurrent with a gradient vector of recurrence $\kappa_\alpha = \partial\kappa(x)/\partial x^\alpha$, and let the covariant derivative of its curvature tensor satisfy (31). From (28) we get

$$\mathscr{B}^{\mu}_{\rho\beta\alpha\gamma\delta} = B^{\mu}_{\rho\beta\alpha}(\kappa_{\gamma;\delta} - \kappa_{\delta;\gamma}) + B^{\mu}_{\rho\gamma\delta}(\kappa_{\beta;\alpha} - \kappa_{\alpha;\beta}) = 0.$$

Because of this equation and (31), it follows at once from (8), (9), (25), (26), and (27) that the restricted extension of a recurrent A^n is recurrent. Also the first n components of its vector of recurrence are the components of the vector of recurrence of A^n while the last n components are zero. Thus we have

THEOREM 3. *If the Riemann extension of A^n is recurrent, then A^n is recurrent, its vector of recurrence is a gradient, and its curvature tensor satisfies* (31). *Conversely, if A^n is recurrent with a gradient vector of recurrence and if the curvature tensor of A^n satisfies* (31), *then the restricted extension of A^n is recurrent.*

† A recurrent A^n has been defined by Wong (**9**). It is to be understood that in the present paper A^n is not flat.

7. Ricci-recurrent spaces

A Riemannian space of more than two dimensions is called *Ricci-recurrent* [Patterson (**3**)] if its Ricci tensor satisfies

$$R_{ij} \neq 0, \qquad R_{ij,k} = R_{ij}\kappa_k$$

for some non-zero vector κ_k. It can be verified that for the Riemann extension of A^n the components of $R_{ij,k}$ are given by

$$R_{\alpha\beta,\rho} = 2b_{\alpha\beta;\rho}, \qquad R_{\alpha\beta,\rho'} = 0, \qquad R_{\alpha' i,j} = 0. \tag{32}$$

Hence, if R^{2n} is Ricci-recurrent, then from (22) and (32),

$$b_{\alpha\beta;\rho} = b_{\alpha\beta}\kappa_\rho, \tag{33}$$

where κ_i ($\kappa_{\alpha'} = 0$) is the vector of recurrence of the Ricci tensor. Consequently we say that A^n is *Ricci-recurrent* if the symmetric part $b_{\alpha\beta}$ of its contracted curvature tensor $B_{\alpha\beta}$ satisfies

$$b_{\alpha\beta} \neq 0, \qquad b_{\alpha\beta;\rho} = b_{\alpha\beta}\kappa_\rho$$

for some non-zero vector κ_ρ.

From (22), (32), and (33) we have

THEOREM 4. *The Riemann extension of A^n is Ricci-recurrent if and only if A^n is Ricci-recurrent.*

It should be noted that in this theorem the Riemann extension is not restricted. We have also, as a corollary, that the unrestricted Riemann extension of a recurrent A^n is Ricci-recurrent; the converse is not however true.

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